Calculation of Inside and Outside Weights of Weighted Hypergraphs by Newton's Method

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Motivation

- Calculation of the corpus likelihood.
- Inside weight is needed in the E-step of the EM-algorithm
- Normalization, transformation of pCFGs [2]

Motivation

Task

Preliminaries

- Hypergraphs
- Inside and outside weights

Approximation

- Fixed-point iteration
- Newton's method
- Decomposed Newton's method

Performance

Task

- Calculation of **Inside and Outside weights** in weighted hypergraphs

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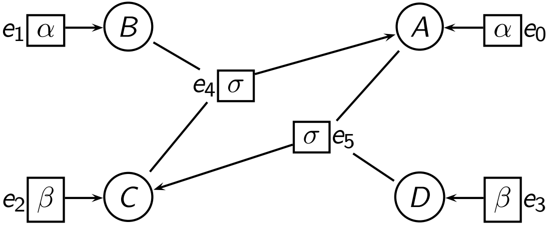
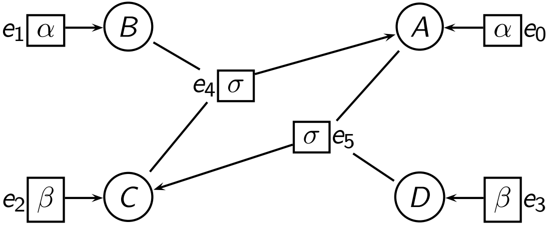
- embedded in **Vanda**

Hypergraphs



- weighted Σ-hypergraph $H = (V, E, \mu)$
- vertices V
- edges $E \subseteq V^* \times \Sigma \times V$
- weights $\mu \colon E \to \mathbb{R}_{\geq 0}$

Hypergraphs



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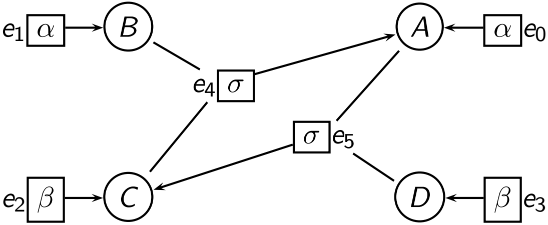

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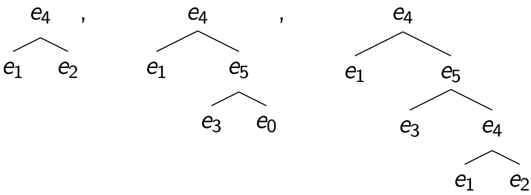
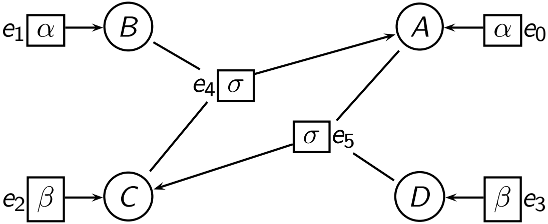


$\in \mathrm{HP}(H, A)$.

Hypergraphs



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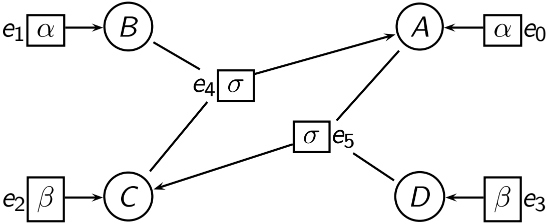

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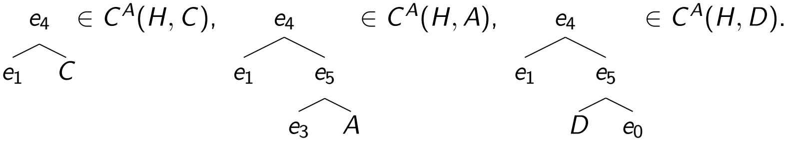

Inside and outside weights

inside weight for a vertex v: sum of the weights of all hyperpaths in H leading to v

$$\mathrm{inside}(v) = \sum_{t \in \mathrm{HP}(H,v)} \mathrm{wt}(t)$$

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outside weight for a vertex v' and a target vertex v: sum of the weights of all v-rooted v'-contexts in H

$$\text{outside}^{v}(v') = \sum_{c \in \text{C}^{v}(H,v')} \text{wt}(c)$$

Fixed-point iteration

- Recursive system

$$\text{inner}(v) = \sum_{\substack{e \in E \\ e=(w,\sigma,v)}} \mu(e) \cdot \prod_{i \in [|w|]} \text{inner}(w_i)$$

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$$\text{outer}^{v'}(v) = s(v) + \sum_{\substack{e \in E \\ e=(w,\sigma,\hat{v}) \\ i \in \mathbb{N}:\ w_i = v}} \text{outer}^{v'}(\hat{v}) \cdot \mu(e) \cdot \prod_{\substack{j \in [|w|] \\ j \neq i}} \text{inner}(w_j)$$

$$s(v) = \begin{cases} 1 & \text{if } v = v' \\ 0 & \text{otherwise} \end{cases}$$

Newton's method

- inside and outside weights as root

$$0 = -i_v + \sum_{\substack{e \in E \\ e=(w,\sigma,v)}} \mu(e) \cdot \prod_{i \in [|w|]} i_{w_i}$$

- uses multivariate Newton's method

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Univariate Newton's method (1 unknown)

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3. If $x_{n+1} = x_n$, output x_n.

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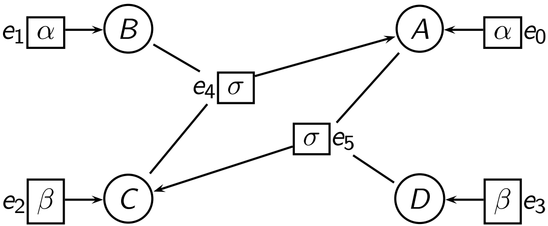
Inside weights as root of a function



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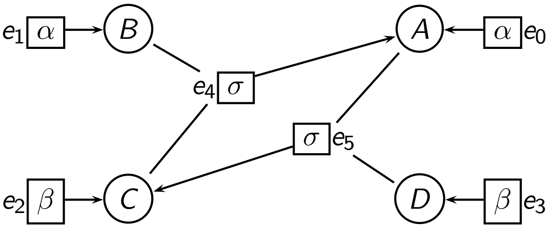

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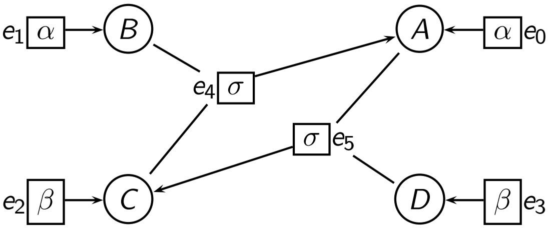

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$$in'(i) = \begin{pmatrix} -i_A + \mu(e_0) + \mu(e_4) \cdot i_C \cdot i_B \\ -i_B + \mu(e_1) \\ -i_C + \mu(e_2) + \mu(e_5) \cdot i_D \cdot i_A \\ -i_D + \mu(e_3) \end{pmatrix}$$

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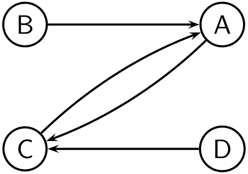

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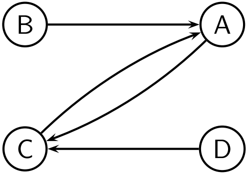

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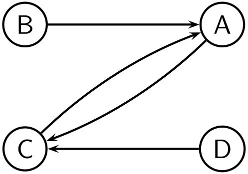

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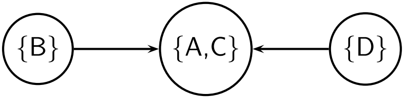

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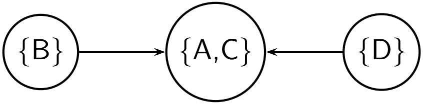

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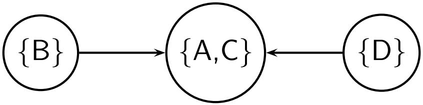
   SCCs $= \langle \{B\}, \{D\}, \{A, C\} \rangle$



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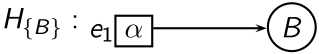

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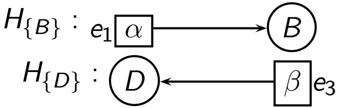

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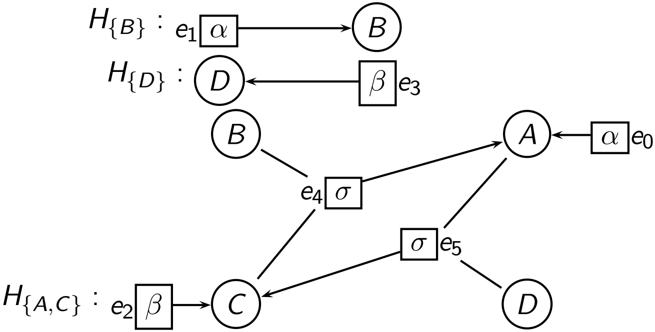

Decomposed Newton's method

4. (Compute polynomials.)
5. Substitute known inside weights in the polynomials with values.
6. (Apply Newton's method.)

For the hypergraph $H_{\{B\}}$ we get

$$\begin{aligned} in'_{\{B\}} &= \left(-i_B + \mu(e_1)\right) \\ &= \left(1 - i_B\right) \\ (Jin'_{\{B\}})^{-1} &= \left(-1\right). \end{aligned}$$

Then by Newton's method in:

$$i_B = 1.$$

Decomposed Newton's method

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For the hypergraph $H_{\{D\}}$ we get

$$\begin{aligned} in'_{\{D\}} &= \left(-i_D + \mu(e_3)\right) \\ &= \left(1 - i_D\right) \\ (Jin'_{\{D\}})^{-1} &= \left(-1\right). \end{aligned}$$

Then by Newton's method:

$$i_D = 1.$$

Decomposed Newton's method

4. (Compute polynomials.)
5. Substitute known inside weights in the polynomials with values.
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For the hypergraph $H_{\{A,C\}}$ we get

$$in'_{\{A,C\}} = \begin{pmatrix} -i_A + \mu(e_0) + \mu(e_4) \cdot i_C \cdot i_B \\ -i_C + \mu(e_2) + \mu(e_5) \cdot i_D \cdot i_A \end{pmatrix}$$

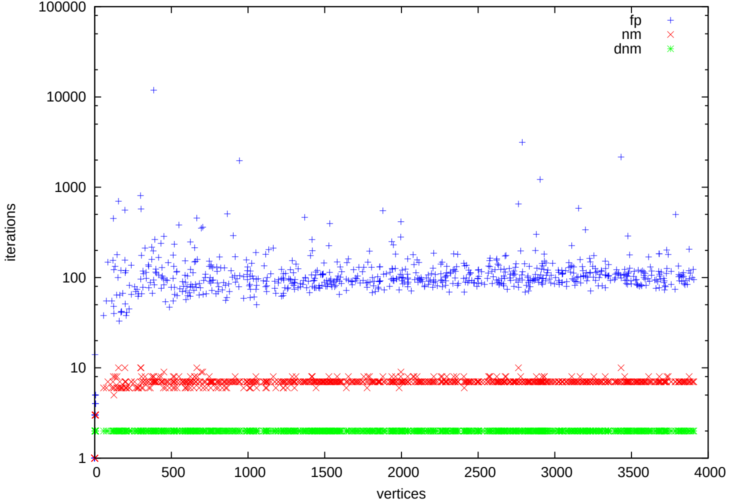
$$= \begin{pmatrix} -i_A + 0.25 + 0.25 \cdot i_C \\ -i_C + 0.25 + 0.25 \cdot i_A \end{pmatrix}$$

$$(Jin'_{\{B\}})^{-1} = -\frac{4}{15} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Then by Newton's method:

$$i_A = \frac{1}{3}, i_C = \frac{1}{3}.$$

Performance



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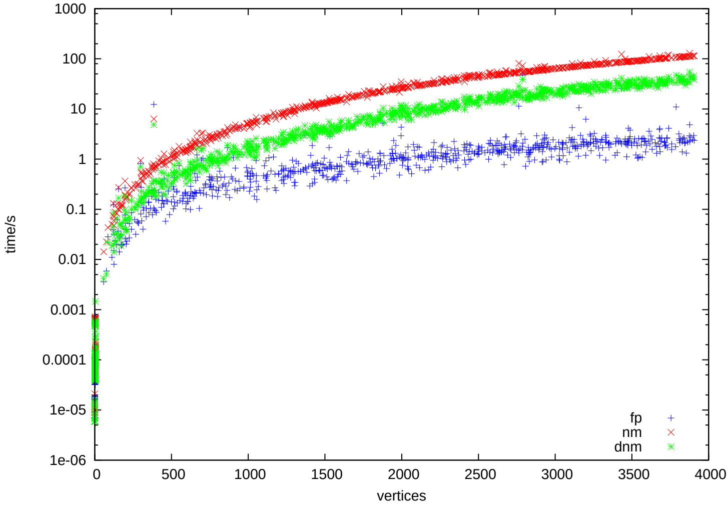
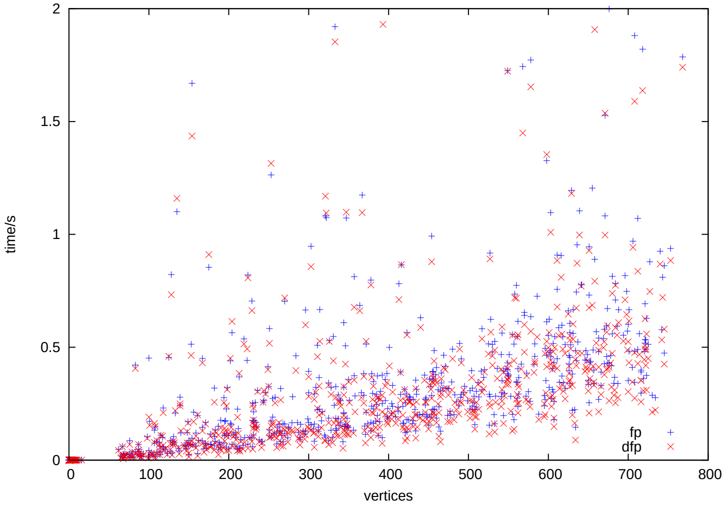


Decomposed fixed-point method



The End

Thank you for your attention!

References

- Javier Esparza, Stefan Kiefer, and Michael Luttenberger, *Convergence thresholds of Newton's method for monotone polynomial equations*, Proceedings of the 25th International Symposium on Theoretical Aspects of Computer Science (STACS) (Bordeaux, France) (Pascal Weil and Susanne Albers, eds.), 2008, http://arxiv.org/abs/0802.2856, pp. 289–300.
- Mark-Jan Nederhof and Giorgio Satta, *Computing partition functions of pcfgs*, Research on Language and Computation **6** (2008), 139–162, 10.1007/s11168-008-9052-8, http://dx.doi.org/10.1007/s11168-008-9052-8.
- Robert Tarjan, *Depth-first search and linear graph algorithms*, SIAM Journal on Computing **1** (1972), no. 2, 146–160, http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=45