A Chomsky-Schützenberger representation for weighted multiple context-free languages

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Chomsky-Schützenberger for CFL & some generalisations

Theorem (Chomsky and Schützenberger 1963, Proposition 2)

Let $L \subseteq \Sigma^*$; the following are equivalent

(i) L is context-free.

(ii) There are

- a homomorphism $h\colon (\Delta \cup \bar{\Delta})^* \to \Sigma^*$,
- a Dyck language $D \subseteq (\Delta \cup \bar{\Delta})^*$, and
- a regular language $R \subseteq (\Delta \cup \bar{\Delta})^*$

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- for multiple context-free languages weighted over complete commutative strong bimonoids (new)

Outline

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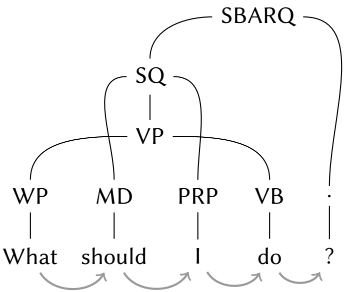
Non-projective trees

Example (Evang and Kallmeyer 2011)



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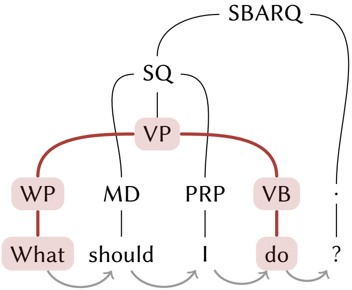

gaps / crossing edges

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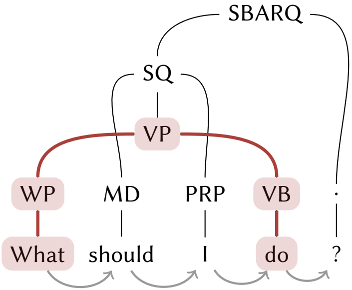

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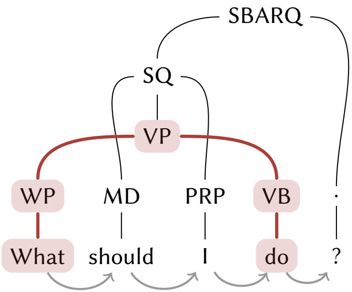

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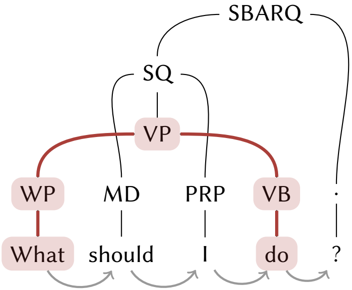

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	proj.	non-proj.
NeGra[1]	72.44%	27.56%
TIGER[2]	72.46%	27.54%

[1] approx. 20 000 trees
[2] approx. 50 000 trees

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a k-ary composition function over Σ

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for some $u_1, \ldots, u_m \in (\Sigma \cup X)^*$

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$$[x_1 y_1, \beta y_2]_{1,2}\colon (\Sigma^*) \times (\Sigma^* \times \Sigma^*) \to (\Sigma^* \times \Sigma^*)$$

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multiple context-free grammar

$$G = (\underbrace{\{S, A, B\}}_{\text{nonterminals}}, \underbrace{\{a, b, c, d\}}_{\text{terminals}}, \underbrace{\{S\}}_{\text{initial nts}}, \underbrace{\{\rho_1, ..., \rho_5\}}_{\text{productions}})$$

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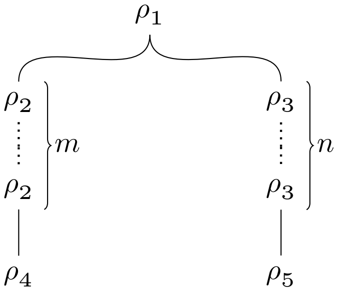

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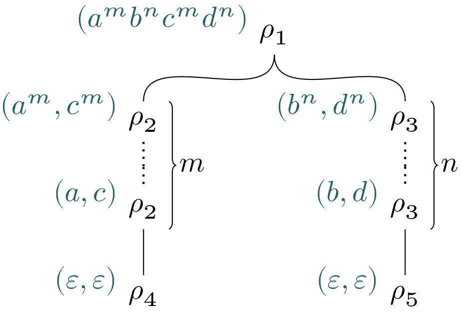

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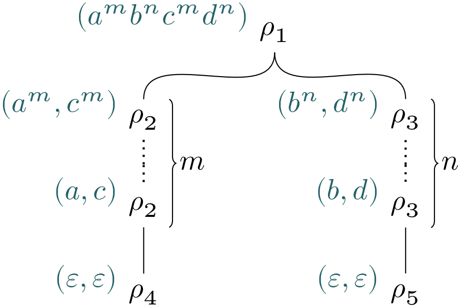

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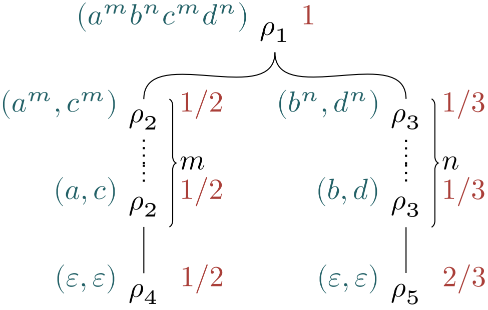

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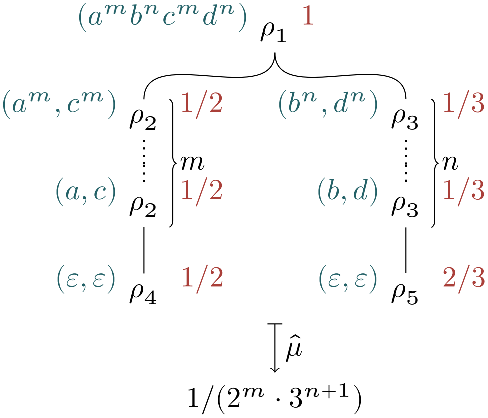

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  - tropical bimonoid: $(\mathbb{R}^{\infty}_{\geq 0}, +, \min, 0, \infty)$

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1. Chomsky-Schützenberger for CFL and some generalisations
2. Non-projective trees and weighted MCFL
3. Parentheses Languages
   - Dyck languages
   - Congruence multiple Dyck language
4. Chomsky-Schützenberger for weighted MCFL
5. Conclusion

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- CFG for $\mathrm{D}(\varDelta)$:

$$A \rightarrow \varepsilon + aA\bar{a} + bA\bar{b} + cA\bar{c} + dA\bar{d} + AA$$

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Congruence multiple Dyck language I

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cf. multiple Dyck languages (Yoshinaka, Kaji, and Seki 2010, Definition 1) for all the details

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Outline

1. Chomsky-Schützenberger for CFL and some generalisations
2. Non-projective trees and weighted MCFL
3. Parentheses Languages
4. Chomsky-Schützenberger for weighted MCFL
   - Weight separation
   - Chomsky-Schützenberger for unweighted MCFL
   - Composing the homomorphisms
5. Conclusion

Chomsky-Schützenberger for weighted MCFL

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(ii) There are

- an $\mathcal{A}$-weighted α-hom. $h\colon (\Delta \cup \bar{\Delta})^* \to \mathcal{A}^{\Sigma^*}$,
- a congruence k-multiple Dyck language $D \subseteq (\Delta \cup \bar{\Delta})^*$, and
- a regular language $R \subseteq (\Delta \cup \bar{\Delta})^*$

such that $\quad L = h(D \cap R)$.

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(ii) $\Rightarrow$ (i) $D \in k\text{-MCF}$ and closure properties of $k\text{-MCF}(\mathcal{A})$

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Weight separation

Lemma (idea from Droste and Vogler 2013, Lemma 3)

For every $\mathcal{A}$-weighted k-MCFL $L\colon \Sigma^* \to \mathcal{A}$ there are an $\mathcal{A}$-weighted α-hom. $h_1\colon (\Sigma \cup R)^* \to \mathcal{A}^{\Sigma^*}$ and an unweighted k-MCFL $L' \subseteq (\Sigma \cup R)^*$ s.t. $L = h_1(L')$.

Weight separation

Lemma (idea from Droste and Vogler 2013, Lemma 3)

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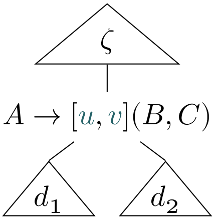

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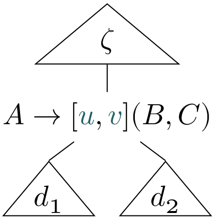

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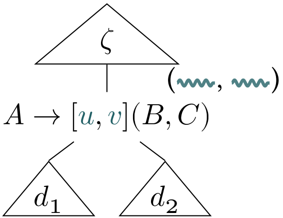

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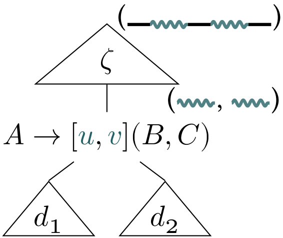

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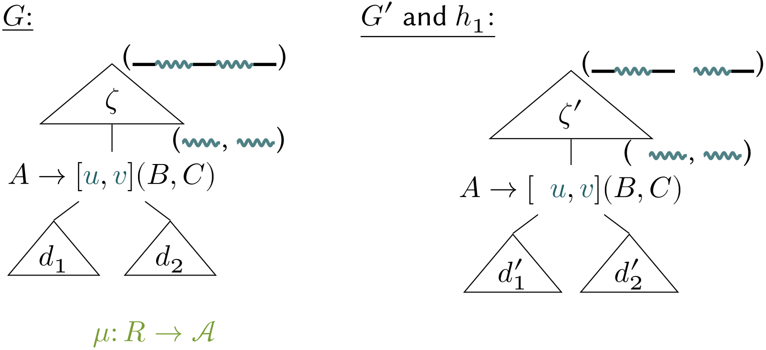

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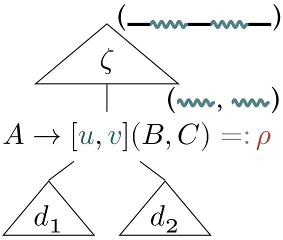

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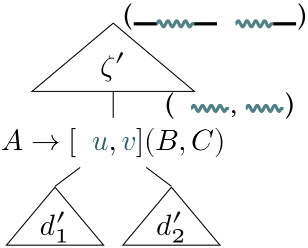

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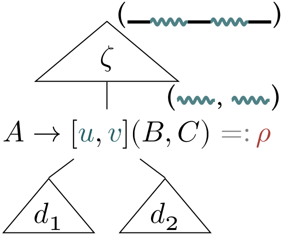

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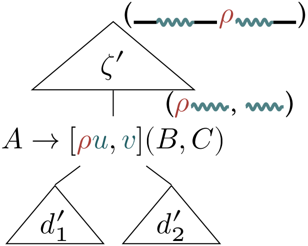

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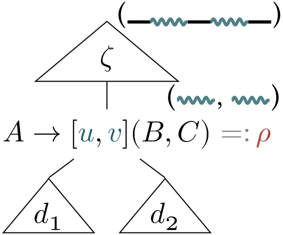

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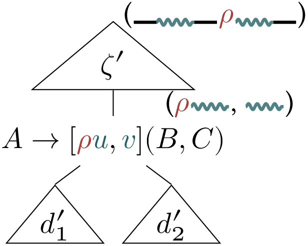

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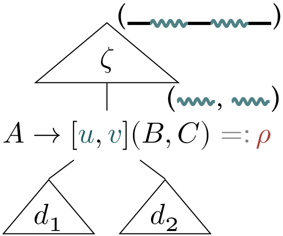

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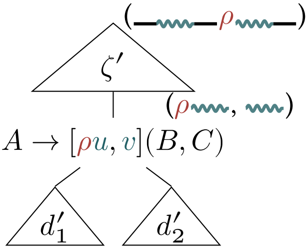

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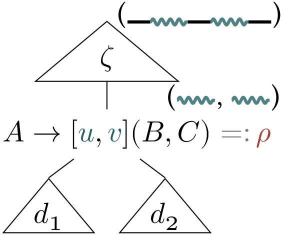

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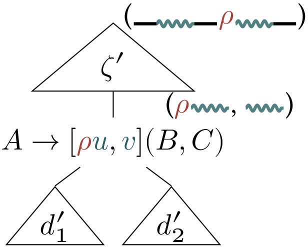

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Chomsky-Schützenberger for weighted MCFL

Theorem

Let $\mathcal{A}$ be a complete commutative strong bimonoid and $L\colon \Sigma^* \to \mathcal{A}$.
For every $k \in \mathbb{N}$, the following are equivalent

(i) L is k-multiple context-free.

(ii) There are

- an $\mathcal{A}$-weighted α-hom. $h\colon (\Delta \cup \bar{\Delta})^* \to \mathcal{A}^{\Sigma^*}$,
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such that $L = h(D \cap R)$.

(ii) ⇒ (i) $D \in k\text{-MCF}$ and closure properties of $k\text{-MCF}(\mathcal{A})$

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Theorem (Yoshinaka, Kaji, and Seki 2010, Theorem 3)

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Conclusion

- definition of multiple Dyck languages using congruence relations
- weight separation for multiple context-free grammars
- Chomsky-Schützenberger result for multiple context-free languages weighted with *complete commutative strong bimonoids*

References

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