The Chomsky-Schützenberger Theorem for Weighted Automata with Storage

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Theorem (CS-Theorem) [Chomsky, Schützenberger 63]

$$L(G) = h(D_n \cap R)$$

G context-free grammar

h homomorphism

D_n n-Dyck language

R regular language

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CS-Theorem for multiple context-free languages

[Fratani, Voundy 15]:

CS-Theorem for indexed languages

weighted language: $L\colon \Sigma^* \to K$ for some weight algebra K
(power series)

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CS-Theorem for weighted iterated pushdown languages

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CS-Theorem for weighted automata with storage

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Unital Valuation Monoid

$(K, +, \mathrm{val}, 0, 1)$ [Droste, Vogler 13]

- $(K, +, 0)$ commutative monoid
- $\mathrm{val}\colon K^* \to K$
  - $\mathrm{val}(a) = a$
  - $\mathrm{val}(\dots 0 \dots) = 0$
  - $\mathrm{val}(\dots 1 \dots) = \mathrm{val}(\dots\dots)$ ignore “1”s
  - $\mathrm{val}(\varepsilon) = 1$

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Examples

1) average

2) strong bimonoid $(K, +, \cdot, 0, 1)$
   - $(K, +, 0)$ commutative monoid
   - $(K, \cdot, 1)$ monoid
   - 0 absorbing for $\cdot$

   e.g., semirings, bounded lattices

Storage [Scott 67]

Storage

[Scott 67]

storage type $S = (C, P, F, C_0)$

- C set (configurations)
- P set of functions $p \colon C \to \{\text{true}, \text{false}\}$ (predicates)
- F set of partial functions $f \colon C \to C$ (instructions)
- C_0 subset of C (initial configurations)

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- TRIV $= (\{c\}, \{p_{\mathrm{true}}\}, \{f_{\mathrm{id}}\}, \{c\})$

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$$p_{\text{true}}(c) = \text{true}$$

$$f_{\text{id}}(c) = c$$

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$$\text{bottom}\left(\boxed{\delta}\boxed{c} \right) = \text{true}$$

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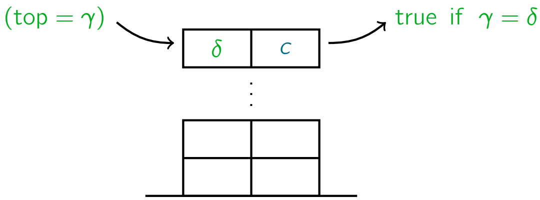

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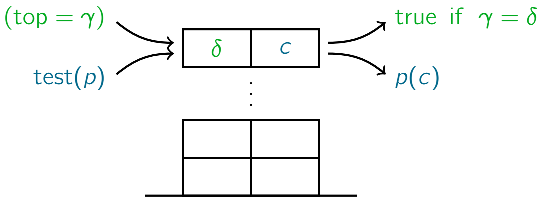

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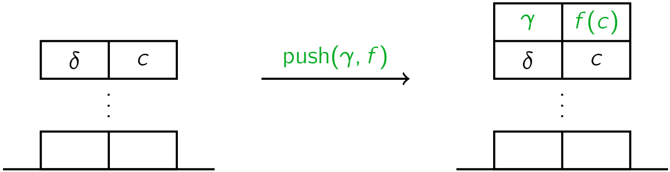

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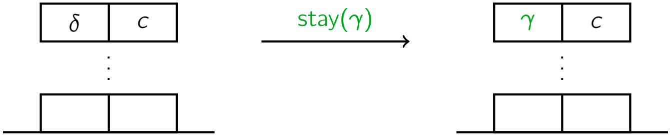

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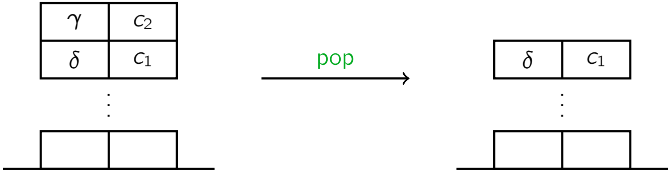

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- iterated pushdown storage [Engelfriet 86/14; Engelfriet, Vogler 86]
  - $\mathrm{P}^0 = \mathrm{TRIV}$
  - $\mathrm{P}^{n+1} = \mathrm{P}(\mathrm{P}^n)$

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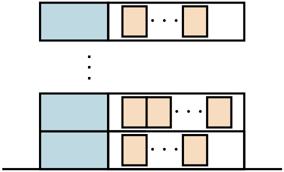

Weighted Automata with Storage

$S = (C, P, F, C_0)$ storage type
Σ alphabet
K unital valuation monoid

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- $c_0 \in C_0$
- T finite set (transitions)

$$\tau = (q_1, x, p, q_2, f)$$

$q_1, q_2 \in Q, \quad x \in \Sigma \cup \{\varepsilon\}, \quad p \in P, \quad f \in F$

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- $\mathrm{wt}\colon T \to K$ (weight assignment)

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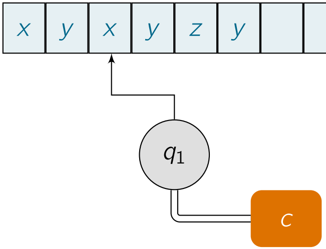

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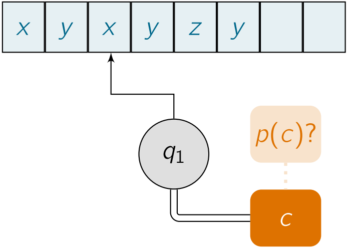

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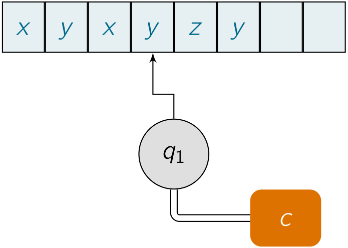

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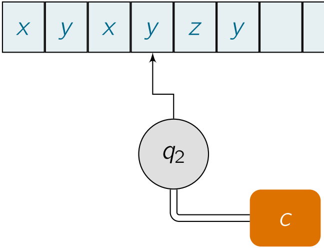

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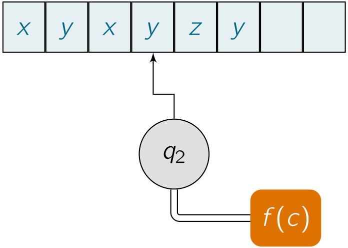

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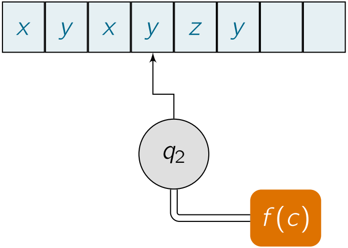

computation step with τ

$$(q_1, xw, c) \vdash^{\tau} (q_2, w, f(c))$$

if $p(c) = \mathrm{true}$
and $f(c)$ defined

Weighted Automata with Storage

$w = x_1 \dots x_n \in \Sigma^*$
$\theta = \tau_1 \dots \tau_n, \quad \tau_i \in T$
$q_f \in Q_f$

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computation:

$$(q_0, x_1 x_2 \ldots x_n, c_0) \vdash^{\tau_1} (q_1, x_2 \ldots x_n, c_1) \vdash^{\tau_2} \ldots \vdash^{\tau_n} (q_f, \varepsilon, c_n)$$

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$\boxed{\Theta(w)}$

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$\boxed{\Theta(w)}$

weighted language recognized by A: $\|A\|\colon \Sigma^* \to K$

$$\|A\|(w) = \sum_{\theta \in \Theta(w)} \mathrm{wt}(\theta)$$

appropriate finiteness condition!

$$\mathrm{wt}(\tau_1 \dots \tau_n) = \mathrm{val}(\mathrm{wt}(\tau_1) \dots \mathrm{wt}(\tau_n))$$

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$\boxed{\mathrm{REC}(S, \Sigma, K)}$

Examples for $\mathrm{REC}(S, \Sigma, K)$

- $\mathrm{REC}(\mathrm{TRIV}, \Sigma, \mathbb{B})$

Examples for REC(S, Σ, K)

- REC(TRIV, Σ, $\mathbb{B}$) TRIV $= (\{c\}, \{p_{\text{true}}\}, \{f_{\text{id}}\}, \{c\})$

$$(q_0, x_1 x_2 \ldots x_n, c) \vdash^{\tau_1} (q_1, x_2 \ldots x_n, c) \vdash^{\tau_2} \ldots \vdash^{\tau_n} (q_n, \varepsilon, c)$$

Examples for $\mathrm{REC}(S, \Sigma, K)$

- $\mathrm{REC}(\mathrm{TRIV}, \Sigma, \mathbb{B})$ $\qquad$ $\mathrm{TRIV} = (\{c\}, \{p_{\mathrm{true}}\}, \{f_{\mathrm{id}}\}, \{c\})$

$$(q_0, x_1 x_2 \ldots x_n \quad) \vdash^{\tau_1} (q_1, x_2 \ldots x_n \quad) \vdash^{\tau_2} \ldots \vdash^{\tau_n} (q_n, \varepsilon \quad)$$

Examples for $\mathrm{REC}(S, \Sigma, K)$

- $\mathrm{REC}(\mathrm{TRIV}, \Sigma, \mathbb{B})$ recognizable languages

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Examples for REC(S, Σ, K)

- REC($\mathrm{TRIV}, \Sigma, \mathbb{B}$) recognizable languages
- REC(P^1, Σ, K) quantitative CF languages over Σ and K

[Droste, Vogler 13]

$$(q_0, x_1 x_2 \ldots x_n, \square) \vdash^{\tau_1} (q_1, x_2 \ldots x_n, \boxminus) \vdash^{\tau_2} \ldots \vdash^{\tau_n} (q_n, \varepsilon, \begin{smallmatrix}\square\\ \vdots \\ \square\end{smallmatrix})$$

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  [Maslov 76; Engelfriet 83; Damm, Goerdt 86]

Examples for $\mathrm{REC}(S, \Sigma, K)$

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Let $L\colon \Sigma^* \to K$.

L is a weighted iterated pushdown language if

$$L \in \bigcup_n \mathrm{REC}(\mathrm{P}^n, \Sigma, K)$$

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- separating weights

$$h : \Delta \to K[\Sigma \cup \{\varepsilon\}]$$
$$h(\delta) = a.y \qquad y \in \Sigma \cup \{\varepsilon\}$$

ε-free $(S, \Delta, \mathbb{B})$-automaton $\mathcal{A}$

$$\Delta^* \supseteq L(\mathcal{A})$$
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$$r\colon \Sigma^* \to K$$

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$\mathcal{M}$ generalized sequential machine (GSM)

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Separating Storage

$$(q_0, ab, c_0) \; \underset{p_1(c_0) = \text{true}}{\overset{(q_0, a, p_1, q_1, f_1)}{\longmapsto}} \; (q_1, b, f_1(c_0)) \; \underset{p_2(f_1(c_0)) = \text{true}}{\overset{(q_1, b, p_2, q_2, f_2)}{\longmapsto}} \; (q_2, \varepsilon, f_2(f_1(c_0)))$$

Separating Storage

of $(S, \Delta, \mathbb{B})$-automaton $\mathcal{A}$

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$(p_1, f_1)(p_2, f_2) \in (P_f \times F_f)^*$

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$$\begin{tikzcd} q_0 \arrow[r, bend left, "(p_1, f_1) \,/\, a"] & q_1 \arrow[r, bend left, "(p_2, f_2) \,/\, b"] & q_2 \end{tikzcd}$$

$$\boxed{\text{GSM } \mathcal{M}}$$

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Instantiated CS-Theorem for (S, Σ, K)-recognizable languages

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- $S = \mathrm{NS}(\mathrm{TRIV})$: K-weighted nested stack automata [Engelfriet, Vogler 86]
- $S = \mathrm{MON}(M)$: K-weighted M-automata for some monoid M [Kambites 07]

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Separating Weights

"$\Rightarrow$"

$\mathcal{B} = (Q, \Sigma, c_0, q_0, Q_f, T, \mathrm{wt})$ $\rightsquigarrow$ $\mathcal{A} = (Q, T, c_0, q_0, Q_f, T')$

$\tau = (q, x, p, q', f)$ in T $\qquad$ $\tau' = (q, \tau, p, q', f)$ in T'

$h(\tau') = \mathrm{wt}(\tau).x$

"$\Leftarrow$"

$\mathcal{A} = (Q, \Delta, c_0, q_0, Q_f, T)$ $\rightsquigarrow$ $\mathcal{B} = (Q', \Sigma, c_0, q_0', \Delta \times Q_f, T', \mathrm{wt})$

$h\colon \Delta \to K[\Sigma \cup \{\varepsilon\}]$ $\qquad$ $Q' = \{q_0'\} \cup \Delta \times Q$

(q, δ, p, q', f) in T $\qquad$ $\tau = ((\delta', q), y, p, (\delta, q'), f)$ in T'

$h(\delta) = a.y$ $\qquad$ $\mathrm{wt}(\tau) = a$

Decomposition of GSM

gsm $\mathcal{M}$	$\leadsto$	regular grammar R, letter-to-letter morphisms h_1, h_2
$\tau = (q, x, q', y)$		$q \to \tau q'$ rule in R
		$h_1(\tau) = x$
		$h_2(\tau) = y$