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Diploma Thesis

Ratio and Weight Objectives in Annotated Markov Chains

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Abstract

In this thesis we investigate the analysis of energy-aware systems modeled by finite-state Markov chains. We deal with Markov chains augmented with transition weights and consider objectives referring to the accumulated weight along finite paths or to the quotient of the accumulated weight of two non-negative weight functions. Beside objectives asking for linear-time properties to be fulfilled while always exceeding a threshold on the accumulated weight or ratio, we allow for properties where the threshold has not to be exceeded globally, but at least once, infinitely often or globally after a finite initialization phase. For these objectives, we aim to exactly compute optimal thresholds such that the corresponding objective is either fulfilled almost-surely or with positive probability (quantiles). To this end, we first state a polynomial-time procedure to decide, whether a weight objective is fulfilled almost-surely or with positive probability. Employing a simple transformation, decision problems referring to ratio objectives are known to be reducible in polynomial time to decision problems related to weight objectives. It turns out that this transformation cannot be applied for quantiles. However, using two different approaches we state polynomial-time computation schemes for both, ratio and weight quantiles. Weight quantiles can be computed by solving the corresponding decision problem within a simple binary search. The algorithm for ratio quantiles relies on establishing a finite set of rational-valued candidates and solving a best-approximation problem using the so-called continued-fraction method.

Acknowledgements

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Contents

1 Introduction

Verification of systems with randomized behaviour or containing uncertainties is gaining more and more interest in terms of quantitative analysis for e.g., energy-aware systems [19, 5, 4, 3]. Uncertainties might arise from only partial knowledge about components and the environment, such as the work load or failure rates (bit flips, ...). One common aim for such systems is computing the probability of a given failure event.

Markovian models and probabilistic model checking are widely used in this context. If amended by rewards, probabilistic model checking for Markovian models additionally allows to reason for example over the consumed energy or gained utility. If one allows for multiple cost functions one could analyze, e.g. the gained utility per consumed energy, short energy-utility ratio, for such a system [5, 4, 3]. Interesting questions in this context are for example:

- What is the probability that the system's energy-utility ratio never drops below a given limit?
- What is the probability that the system reaches a configuration where the energy-utility never drops below a given limit?
- What is the probability that eventually the system's energy-utility ratio is higher than a given limit?

Further, if we allow for positive and negative rewards, one could model a system charging and draining a (possibly unbounded) battery. Given an initial battery level, we could ask for the probability that the system runs out of battery. Another interesting question would be:

- What is the lowest initial battery level to ensure that system does not run out of battery?

This question is closely related to the concept of quantiles, which is well-known from statistics (see e.g. [24]), statistical analysis and performance prediction. Given a random variable R and a probability threshold p, a *p-quantile* asks for the maximal value r such that $Pr(R > r) > p$ or $Pr(R > r) \geq p$. Qualitative quantiles are a special case of p-quantiles, where we only ask for the maximal value r such that the constraint $R > r$ is satisfied almost-surely or with positive probability, i.e., $Pr(R > r) = 1$ or $Pr(R > r) > 0$. Almost-sure quantiles enable to analyze the worst-case performance of a system, whereas positive quantiles consider the best-case. The above question is in fact (indirectly) asking for a positive quantile: What is the highest initial battery level such that the probability of running out of battery is greater than 0?

Let us consider another application of qualitative quantiles. Assume we can choose between two system variants applicable for a certain task, which differ in both, their price and performance. If the performane can be modeled by an energy-utility ratio, one way to compute the trade-off of using the more expensive system would be to compute and compare the qualitative energy-utility quantiles

$$\sup\{q \in \mathbb{Q} : \; Pr(E) > 0\} \qquad \text{and} \qquad \sup\{q \in \mathbb{Q} : \; Pr(E) = 1\},$$

where E contains executions satisfying a certain energy-utility related objective, e.g.,

$$E = \left(\text{all executions, where } \frac{\text{utility}}{\text{energy}} \text{ eventually stays greater than } q\right) \text{ or}$$
$$E = \left(\text{all executions, where } \frac{\text{utility}}{\text{energy}} \text{ is always than } q\right).$$

Beside being interesting themself, these quantiles yield other indicators such as the absolute deviations δ_a or the relative deviation δ_r which can be used as a measure of dispersion. Let $\mathsf{Qu}^{=1}$ be an almost-sure quantile for some event E and $\mathsf{Qu}^{>0}$ the corresponding positive quantile for E, then

$$\delta_a = \frac{1}{2}\left|\mathsf{Qu}^{=1} - \mathsf{Qu}^{>0}\right| \qquad \text{and} \qquad \delta_r = \left|\frac{\mathsf{Qu}^{=1} - \mathsf{Qu}^{>0}}{\mathsf{Qu}^{=1} + \mathsf{Qu}^{>0}}\right|.$$

High dispersion might indicate great adaptivity of a system, whereas low dispersion may imply stability and robustness.

Scope and Contribution. In this thesis we consider Markov chains extended with one or two reward functions. We investigate decision problems and quantiles for objectives related to these reward functions and their ratios. The main contribution of this thesis is to that the following quantles are computable in polynomial time:

$$\begin{array}{rl} \text{weight quantiles:} & \sup\{z \in \mathbb{Z} : Pr(\heartsuit(\mathsf{weight} > z) \wedge \phi) > 0\}, \\ & \sup\{z \in \mathbb{Z} : Pr(\heartsuit(\mathsf{weight} > z) \wedge \phi) = 1\}, \\ \text{ratio quantiles:} & \sup\{q \in \mathbb{Q} : Pr(\heartsuit(\mathsf{ratio} > q) \wedge \phi) > 0\}, \\ & \sup\{q \in \mathbb{Q} : Pr(\heartsuit(\mathsf{ratio} > q) \wedge \phi) = 1\}, \end{array}$$

where $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$, ratio denotes the quotient $\mathsf{utility}/\mathsf{energy}$ and ϕ is an omega-regular property encoded by a deterministic Rabin automaton. The event $(\Box(\mathsf{weight} > z) \wedge \phi)$ contains all paths satisfying ϕ and where the accumulated weight is always greater than z. For $\Diamond$ (eventually), $\Diamond\Box$ (from some point onwards), $\Box\Diamond$ (infinitely often) and energy-utility ratios, the corresponding events are defined analogously. Whereas energy and $\mathsf{utility}$ range over $\mathbb{N}$, we allow weight to contain both negative and positive integers.

The presented methods rely on a polynomial decision procedure for

$$Pr(\heartsuit(\mathsf{weight} > z) \wedge \phi) > 0, \qquad Pr(\heartsuit(\mathsf{weight} > z) \wedge \phi) = 1,$$
$$Pr(\heartsuit(\frac{\mathsf{utility}}{\mathsf{energy}} > q) \wedge \phi) > 0, \qquad Pr(\heartsuit(\frac{\mathsf{utility}}{\mathsf{energy}} > q) \wedge \phi) = 1.$$

Whereas the proof that the above decision problems are solvable in polynomial time shares some ideas with existing results for unary cost functions, i.e., cost functions ranging over $\{-1, 0, +1\}$ instead of $\mathbb{N}$ [10, 9], the investigated weight and ratio quantiles have not yet been considered. The results of this thesis have been submitted to the *40th International Symposium on Mathematical Foundations of Computer Science 2015* [22].

Related Work. A variety of state-based models using a single weight function has been studied in the literature. However, most of them require the weights to be contained in $\{+1, 0, -1\}$.

An example of such a model are *probabilistic pushdown machines* (pPDA) with exactly one stack symbol, also referred to as one-counter automata. It is known that one can decide using a polynomial-space algorithm whether for a given pPDA and a given probability threshold p a specific configuration is reachable with probability greater than p. See [8] for a survey on pPDA. If restricted to one-counter automata, one can decide in polynomial time whether almost-surely the counter stays positive [10]. In [13] Etessami, Wojtczak, and Yannakakis show that the probability of the following event can be approximated up to a precision ε in polynomial time with respect to the size of the counter automaton and $\log(1/\varepsilon)$: all executions, where the counter equals zero when reaching a given state and was never zero before. Additionally the probability of all runs satisfying a given ω-regular property encoded by a deterministic Rabin automaton can be approximated in time polynomial in the sizes of both automata and $\log(1/\varepsilon)$ [7].

One-counter Markov decision processes are another well-studied example for models using a single unary weight function, i.e., ranging over $\{+1, 0, -1\}$. Markov decision processes (MDP) are a state-based model, where the outgoing transition can be chosen either non-deterministically or probabilistically. For one-counter MDPs it is decidable in polynomial time whether given a state and an initial credit there exists a strategy to resolve the non-deterministic choices and ensure that the counter eventually reaches zero almost-surely [9].

Observe, although the above results are closely related to our contribution, unary weight functions are a serious restriction, as transforming a weight function over $\mathbb{Z}$ to a weight function over $\{+1, 0, -1\}$ imposes an exponential blow-up.

In [2] Baier et al. show that if we allow weight functions over $\mathbb{Z}$, it is still decidable in polynomial time, whether all strategies almost-surely ensure the counter to be always positive.

Energy parity games are turn-based two player games on weighted graphs played for infinitely many rounds. In contrast to one-counter MDPs they allow for weight functions over $\mathbb{Z}$. The goal of player 1 is to ensure that both the accumulated weight stays positive throughout the game and the play satisfies a given ω-regular property, where as player 0 tries to sabotage. The *minimal initial credit problem*, which asks for the least credit such that there exists a winning strategy for player 1, is solvable in pseudo-polynomial time [12]. Notice that the minimal initial credit problem is closely related to qualitative weight quantiles.

To the best of our knowledge, the polynomial computation of *qualitative weight and ratio quantiles* has not been considered. The concept of quantiles within Markovian models has been first introduced in [27, 5]. The authors consider the computation of optimal schedulers for quantiles with reachability constraints using iterative linear-programming approaches. Upper complexity bounds for these quantiles are discussed in [18].

Organization. After introducing basic concepts and our notations for Markov chains, linear-time properties and random walks in Chapter 2, we formally define the above mentioned quantiles and decision problems in Chapter 3. In Chapter 4 and Chapter 5 we show that under the assumption of the side-constraint ϕ being negligible, the decision problems and quantiles are solvable in polynomial time. We then generalize these results for arbitrary omega-regular properties ϕ in Chapter 6. In Chapter 7 we introduce the modality *strong release* and show that the techniques presented in the previous sections can be amended to the related qualitative decision problems and quantiles. A brief summary of the thesis and future research directions can be found in Chapter 8.

2 Preliminaries

In this chapter we introduce our notations for Markov chains, omega-regular properties and random walks. Further, we will recall some known results. In Section 2.1 we briefly introduce the concept of finite Markov chains and extend it by reward functions in order to define weighted and energy-utility Markov chains. A short introduction to omega-regular properties with respect to Markov chains can be found in Section 2.2, where we also define LTL-style notations for the analysis of weighted and energy-utility Markov chains. Section 2.3 contains a definition of random walks and some basic results which will be of great help in Chapter 4.

We assume that the reader is familiar with basic concepts of probability theory, automata and model checking (see, e.g., [21], [20], [1]).

2.1 Markov chains

Markov chains are widely used to model systems which involve randomized behaviour or contain uncertainties. In this section, we briefly introduce the concept of Markov chains including their induced probability space and bottom strongly connected components. Additionally we extend the standard Markov chain notion by reward functions to define weighted Markov chains and energy-utility Markov chains.

Definition 2.1. *A* Markov chain $\mathcal{M}$ *is a tuple* (S, P, ι, AP, L) *where* S *is a nonempty finite set of* states, $\iota \in S$ *the* initial state *and* $P\colon S \times S \to [0,1]$ *is a* transition probability matrix, *i.e.,* P *satisfies* $\sum_{s' \in S} P(s,s') = 1$ *for all* $s \in S$. *Further,* AP *is a finite set of* atomic propositions *and* L *a labelling function* $L\colon S \to 2^{AP}$.

The size $|\mathcal{M}|$ *of a Markov chain* $\mathcal{M}$ *is defined as the binary encoding length of* P *plus the number of states and transitions, where a* transition *is a pair of states* (s,s') *with* $P(s,s') > 0$ *, i.e.,*

$$|\mathcal{M}| = |S| + |\{(s,s') : \ P(s,s') > 0\}| + elen(P).$$

Basic path notations

Let us fix a Markov chain $\mathcal{M} = (S, P, \iota, AP, L)$. An infinite sequence $s_0 s_1 s_2 \ldots$ of states is called an *infinite path* of $\mathcal{M}$ if each pair of consecutive states is a transition of $\mathcal{M}$. The set of all infinite paths of $\mathcal{M}$ is denoted by $\mathsf{InfPaths}^{\mathcal{M}}$. Given a state $s \in S$, $\mathsf{InfPaths}^{\mathcal{M}}_s$ denotes the corresponding set of all infinite paths starting in s.

Analogously, *finite paths* are nonempty finite sequences of states comprising of consecutive transitions. The corresponding sets $\mathsf{FinPaths}^{\mathcal{M}}$ and $\mathsf{FinPaths}^{\mathcal{M}}_s$ are defined as for infinite paths. In general we denote (infinite) paths with π and finite

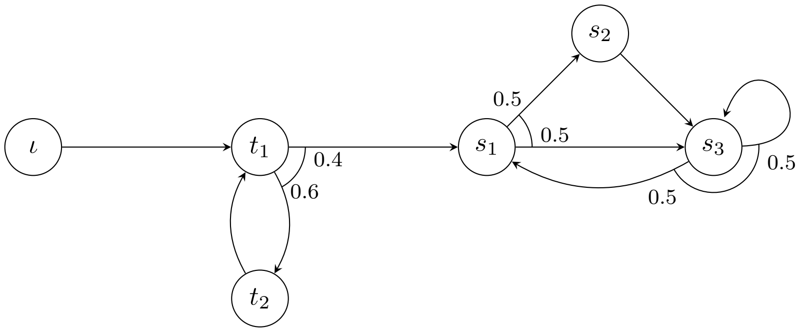

Figure 2.1: Plain Markov chain

paths with $\hat{\pi}$. The *length* $|\hat{\pi}|$ of a finite path $\hat{\pi}$ stands for the number of its transitions, e.g., $|s_0 s_1 s_2| = 2$. The first and last state of a path $\hat{\pi}$ are denoted by $\mathsf{first}(\hat{\pi})$ and $\mathsf{last}(\hat{\pi})$, respectively.

Notice that given two finite paths $\hat{\pi}_1$ and $\hat{\pi}_2$ the sequence $\hat{\pi}_1\hat{\pi}_2$ constitutes a finite path if and only if $\mathsf{last}(\hat{\pi}_1)\hat{\pi}_2$ is a finite path. This motivates the definition of the following concatenation operator.

Definition 2.2. *Let* $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}$ *be a finite path and* $\pi = \mathsf{last}(\hat{\pi})\pi'$ *be a not necessarily finite path contained in* $(\mathsf{FinPaths}^{\mathcal{M}}_{\mathsf{last}(\hat{\pi})} \cup \mathsf{InfPaths}^{\mathcal{M}}_{\mathsf{last}(\hat{\pi})})$*. We define the extension of* $\hat{\pi}$ *by* π *as* $\hat{\pi} \diamond \pi := \hat{\pi}\pi'$*.*

For an infinite path $\pi = s_0 s_1 s_2 \ldots$ and $n \in \mathbb{N}$ we will use the following notions:

$$\pi[n] = s_n \qquad \pi[n\ldots] = s_n s_{n+1}\ldots \qquad \pi[\ldots n] = s_0 s_1 \ldots s_n$$

Observe that for any infinite path $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$ and $n \in \mathbb{N}$ we have $\pi = \pi[\ldots n] \diamond \pi[n\ldots]$.

Let us now consider some important subclasses of finite paths. A *simple path* is a finite path which contains each state at most once. A *cycle* is a finite path ϑ of length at least one starting and ending in the same state, i.e., $\mathsf{first}(\vartheta) = \mathsf{last}(\vartheta)$ and $|\vartheta| \geq 1$. We call a cycle ϑ *simple*, if there exists a simple path $\hat{\pi}$ such that $\vartheta = \hat{\pi}\,\mathsf{first}(\hat{\pi})$.

Example 2.3. Consider the Markov chain $\mathcal{M} = (S, P, \iota, AP, L)$ depicted in Figure 2.1, where $AP = \varnothing$. $\mathcal{M}$ contains exactly four simple cycles, namely $t_1t_2t_1$, $s_1s_2s_3s_1$, $s_1s_3s_1$ and s_3s_3. The sequence $\pi = \iota t_1 t_1 (s_3)^\omega$ is an infinite path contained in $\mathsf{InfPaths}^{\mathcal{M}}_{\iota}$ with

$$\pi[4] = s_3, \qquad \pi[4\ldots] = (s_3)^\omega \qquad \text{and} \qquad \pi[\ldots 4] = \iota t_1 s_1 s_3 s_3.$$

Notice that in Example 2.3 every state of the Markov chain is reachable from ι, which means for every state s that there exists a finite path $\hat{\pi}$ with $\mathsf{first}(\hat{\pi}) = \iota$ and $\mathsf{last}(\hat{\pi}) = s$. Further, the Markov chain does not contain states with no outgoing transition. Whereas the latter is a consequence of our definition of Markov chains, we in the following assume w.l.o.g. that all states of a Markov chain are reachable from ι.

Probability space

Up to now, we have just considered notions related to the underlying graph of a Markov chain. Let us not turn to the probabilistic choice.

Consider again the Markov chain depicted in Figure 2.1. Intuitively, the probability of the finite path $\hat{\pi} = \iota t_1 t_2 s_1$ is $P(\iota, t_1) \cdot P(t_1, t_2) \cdot P(t_2, t_1) \cdot P(t_1, s_1) = 0.6 \cdot 0.4$.

However, as we not only want to consider paths up to a fixed length, we need to define the probability space over the infinite paths of a Markov chain. We formalize the intuitive notion of probabilities as in [1] using the *cylinder sets* spanned by finite paths of $\mathcal{M}$. The cylinder set $Cyl(\hat{\pi})$ for a given finite path $\hat{\pi}$ contains all infinite paths π which start with $\hat{\pi}$, i.e., such that $\pi = \hat{\pi} \diamond \pi'$ for some $\pi' \in \mathsf{InfPaths}^{\mathcal{M}}$.

These cylinder sets constitute a basis of a σ-algebra $\mathfrak{S}$ which we use to define the probability measure $Pr_s^{\mathcal{M}}$ for $\mathcal{M}$ and a given state s of $\mathcal{M}$. Let $Pr_s^{\mathcal{M}}$ be the unique probability measure on $\mathfrak{S}$ such that for every finite path $\hat{\pi} = s_0 \dots s_n \in \mathsf{FinPaths}_s^{\mathcal{M}}$.

$$Pr_s^{\mathcal{M}}(Cyl(s_0 \dots s_n)) = \prod_{i=1}^{n} P(s_{i-1}, s_i)$$

and $Pr_s^{\mathcal{M}}(Cyl(\hat{\pi})) = 0$, otherwise. Existence and uniqueness of $Pr_s^{\mathcal{M}}$ are ensured by Caratheodory's measure-extension theorem see e.g., [21, Theorem A1.1].

Bottom strongly connected components (BSCC)

A bottom strongly connected component (BSCC) can be seen as a "trap" set of states, where if the system once enters a state of this set, it cannot visit any state outside this trap set anymore. Consider the Markov chain depicted in Figure 2.1. Once the sytem has entered state in $\{s_1, s_2, s_3\}$ it can only reach states within this set. To ensure the minimality of the considered sets, we additionally require that the set is strongly connected, i.e., every state in the set is reachable from every other state in the set.

Definition 2.4. *Let $\mathcal{M} = (S, P, \iota, AP, L)$ be a Markov chain. A nonempty subset $\mathcal{C}$ of S is referred to as* bottom strongly connected component (BSCC) *of $\mathcal{M}$ if the following two statements hold:*

- $\mathsf{last}(\hat{\pi}) \in \mathcal{C}$, *for all $s \in \mathcal{C}$ and $\hat{\pi} \in \mathsf{FinPaths}_s^{\mathcal{M}}$*
- *for all $s, s' \in \mathcal{C}$, there exists $\hat{\pi} \in \mathsf{FinPaths}_s^{\mathcal{M}}$ such that* $\mathsf{last}(\hat{\pi}) = s'$

Example 2.5. Consider the Markov chain depicted in Figure 2.1. It contains exactly one BSCC, namely $\mathcal{C} = \{s_1, s_2, s_3\}$.

The BSCCs of $\mathcal{M}$ are computable in time polynomial in the size of $\mathcal{M}$ using e.g., Tarjan's algorithm [25]. The analysis of BSCCs is crucial for the analysis of a given Markov chain, as almost all paths eventually enter a BSCC $\mathcal{C}$ and visit all states of this BSCC $\mathcal{C}$ infinitely often (cf. [1], Theorem 10.27).

Additionally, for almost all paths reaching a BSCC $\mathcal{C}$, the following limit is constant

$$\mathbb{S}_{\mathcal{C}}(s) = \lim_{n\to\infty} \frac{1}{n+1} \;|\{k \in \{0,\ldots,n\} :\; \pi[k] = s\}|$$

for a given state $s \in C$ (cf. Chapter 7 of [21]). In the following we will refer to $\mathbb{S}_{\mathcal{C}}$ as the *steady-state probability* of $\mathcal{C}$. The steady state probability is the solution to the following linear equation system

$$\text{for all } s \in \mathcal{C} : \quad \sum_{s' \in \mathcal{C}} P(s', s) \cdot \mathbb{S}_{\mathcal{C}}(s') = \mathbb{S}_{\mathcal{C}}(s) \qquad \text{and} \qquad \sum_{s' \in \mathcal{C}} \mathbb{S}_{\mathcal{C}}(s') = 1$$

and thus can be computed in polynomial time.

Example 2.6. Reconsider the Markov chain depicted in Figure 2.1. Let the event $E_1 \subseteq \mathsf{InfPaths}_{\iota}^{\mathcal{M}}$ be the set of all paths infinitely often visiting t_1 and $E_2 \subseteq \mathsf{InfPaths}_{\iota}^{\mathcal{M}}$ the set of all paths infinitely often visiting s_1. Even though both sets are nonempty, $\mathcal{C} = \{s_1, s_2, s_3\}$ being the only BSCCs implies

$$Pr_{\iota}^{\mathcal{M}}(E_1) = 0 \qquad \text{and} \qquad Pr_{\iota}^{\mathcal{M}}(E_2) = 1.$$

Further, the steady-state of $\mathcal{C}$ evaluates to:

$$\mathbb{S}_{\mathcal{C}}(s_1) = \frac{2}{7} \qquad \mathbb{S}_{\mathcal{C}}(s_2) = \frac{1}{7} \qquad \mathbb{S}_{\mathcal{C}}(s_3) = \frac{4}{7}$$

Weighted Markov chains

Weighted Markov Chains amend Markov chains by a *weight function* which associates an integer to every transition.

Definition 2.7. *Let $\mathcal{M} = (S, P, \iota, AP, L)$ be a Markov chain. A* weight function *for $\mathcal{M}$ is a function $\mathsf{weight} : S \times S \to \mathbb{Z}$ where $\mathsf{weight}(s, s') = 0$ if $P(s, s') = 0$. A* reward function *for $\mathcal{M}$ is a weight function for $\mathcal{M}$ which maps to $\mathbb{N}$ only.*

Definition 2.8. *Let $\mathcal{M} = (S, P, \iota, AP, L)$ be a Markov chain and let weight be a weight function for $\mathcal{M}$. We refer to $(S, P, \iota, AP, L, \mathsf{weight})$ as a* weighted Markov chain.

The size of a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ is defined as the size of the Markov chain $\mathcal{M}' = (S, P, \iota, AP, L)$ plus the binary encoding length of weight, i.e.,

$$|\mathcal{M}| = |\mathcal{M}'| + \mathit{elen}(\mathsf{weight}).$$

We now lift the notion of weight to paths. To this end, let us fix a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$. The *accumulated weight* $\mathsf{weight}(\hat{\pi})$ of a finite path $\hat{\pi} = s_0\, s_1\, s_2\, \ldots\, s_n$ is the sum of all its transition weights, i.e.,

$$\mathsf{weight}(\hat{\pi}) = \sum_{i=1}^{n} \mathsf{weight}(s_{i-1}, s_i).$$

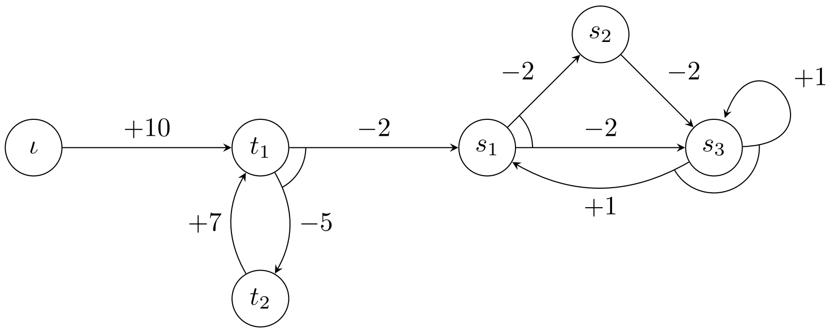

Figure 2.2: Weighted Markov chain (choices are assumed to be uniformly distributed)

Thus, $\mathsf{weight}(\hat{\pi}) = 0$ for every finite path with $|\hat{\pi}| = 0$. If the accumulated weight of a finite path $\hat{\pi}$ is positive, i.e., $\mathsf{weight}(\hat{\pi}) > 0$, $\hat{\pi}$ is said to be weight-*positive* or simply *positive*. Analogously, if $\mathsf{weight}(\hat{\pi}) < 0$ the finite path is weight-*negative* or simply *negative*.

In this thesis will frequently refer to the *minimal weight* $\mathsf{minwgt}[\mathsf{weight}](\hat{\pi})$ on a finite path $\hat{\pi}$. If $|\hat{\pi}| > 0$, it is defined as

$$\mathsf{minwgt}[\mathsf{weight}](\hat{\pi}) = \min_{\rho}(\mathsf{weight}(\rho)),$$

where ρ ranges over all nonempty prefixes of $\hat{\pi}$. Otherwise, $\mathsf{minwgt}[\mathsf{weight}](\hat{\pi}) = \infty$.

Another important notion for weighted Markov chains is the *expected weight for a BSCC $\mathcal{C}$* defined as

$$\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = \sum_{s,s' \in \mathcal{C}} \mathsf{weight}(s, s') \cdot \mathbb{S}_{\mathcal{C}}(s) \cdot P(s, s').$$

The expected weight is computable in polynomial time, as both, the BSCCs of $\mathcal{M}$ and their steady-state probabilities are computable in polynomial time.

Example 2.9. Consider the weighted Markov chain depicted in Figure 2.2, which is essentially a weighted version of the Markov chain depicted in Figure 2.1, but for the distribution in state t_1. With respect to weight, the simple cycle $\vartheta_1 = t_1 t_2 t_1$ constitutes a positive cycle, whereas $\vartheta_2 = s_1 s_2 s_3 s_1$ is a negative cycle, as $\mathsf{weight}(\vartheta_2) = -3$. Further $\mathsf{minwgt}(\vartheta_2) = -4$. As discussed in Example 2.6, the Markov chain contains exactly one BSCC $\mathcal{C} = \{s_1, s_2, s_3\}$ with $\mathbb{S}_{\mathcal{C}}(s_1) = 2/7$, $\mathbb{S}_{\mathcal{C}}(s_2) = 1/7$ and $\mathbb{S}_{\mathcal{C}}(s_3) = 4/7$. Thus, the expected weight of this BSCC is $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = -2/7$.

Energy-utility Markov chains

Energy-utility Markov chains amend Markov chains by two reward functions energy and utility.

Definition 2.10. *Let* $\mathcal{M} = (S, P, \iota, AP, L)$ *be a Markov chain and let* energy*,* $\mathsf{utility}$ *be two reward functions for* $\mathcal{M}$ *such that* energy *is strictly positive, i.e., maps only to* $\mathbb{N}_{>0}$*. We refer to* $(S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ *as an* energy-utility Markov chain*. The* energy-utility ratio*, briefly* ratio*, is defined as* $\mathsf{ratio} : \mathsf{FinPaths}^{\mathcal{M}} \to \mathbb{Q}_{\geq 0}$ *given by* $\mathsf{ratio}(\hat{\pi}) = \mathsf{utility}(\hat{\pi})/\mathsf{energy}(\hat{\pi})$*, where* $|\hat{\pi}| > 0$ *and* $\mathsf{ratio}(\hat{\pi}) = 0$ *otherwise.*

The size of an energy-utility Markov $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ *chain is defined as the size of the Markov chain* $\mathcal{M}' = (S, P, \iota, AP, L)$ *plus the encoding lengths of* energy *and* $\mathsf{utility}$*, i.e.,*

$$|\mathcal{M}| = |\mathcal{M}'| + \mathit{elen}(\mathsf{energy}) + \mathit{elen}(\mathsf{utility}).$$

It is well-known that for almost all paths π eventually reaching a given $\mathcal{C}$ of $\mathcal{M}$ the ratio of all its prefixes converges towards $\mathbb{E}_{\mathcal{C}}(\mathsf{utility})/\mathbb{E}_{\mathcal{C}}(\mathsf{energy})$ [2]. Formally,

$$\lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n]) = \frac{\mathbb{E}_{\mathcal{C}}(\mathsf{utility})}{\mathbb{E}_{\mathcal{C}}(\mathsf{energy})}.$$

In the following we refer to $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = \lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n])$ as the *long-run ratio* of $\mathcal{C}$. Observe that $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio})$ is rational as $\mathbb{S}_{\mathcal{C}}$ being rational implies that both $\mathbb{E}_{\mathcal{C}}(\mathsf{utility})$ and $\mathbb{E}_{\mathcal{C}}(\mathsf{energy})$ rational. Further, $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio})$ is computable in polynomial time.

2.2 Linear-time properties

Linear-time properties are used to specify desirable behaviour of a given system - in our case Markov chains. We will here just give a brief introduction to the field of linear-time properties, limited to the extent required by this thesis. For a comprehensive introduction, see [1].

Let us first consider how to formalize behaviour of a Markov chain. To this end, let us fix $\mathcal{M} = (S, P, \iota, AP, L)$. Assume that the current state of a Markov chain is not observable, but only the atomic propositions assigned to this state. Each execution, represented by an infinite path $\pi = s_0 s_1 s_2 \ldots$ induces an observable behaviour which we refer to as *trace* of π,

$$\mathit{trace}(\pi) = L(s_0)\, L(s_1)\, L(s_2) \ldots .$$

A *linear-time property* ϕ is a set of traces, i.e., a subset of $(2^{AP})^{\omega}$. Model checking the linear-time property ϕ for a given Markov chain $\mathcal{M} = (S, P, \iota, AP, L)$ boils down to computing the following probability

$$Pr_s^{\mathcal{M}} \left\{ \pi \in \mathsf{InfPaths}_s^{\mathcal{M}} : \ \mathit{trace}(\pi) \in \phi \right\}.$$

One way to specify a linear-time property ϕ is to construct an automaton over the alphabet 2^{AP} which accepts exactly all the traces contained in ϕ. Suitable automata for this purpose are for example deterministic Rabin or Streett automata, which are both special types of deterministic ω-automata.

Definition 2.11. *A* deterministic ω-automaton *is a tuple* $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$*, where* Q *is finite set of states,* Σ *is a finite alphabet,* $\delta : Q \times \Sigma \to Q$ *is a deterministic transition relation,* $q_0 \in Q$ *is the initial state and* Acc *an acceptance condition.*

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ be an ω-automaton. An infinite sequence of states $q_0\, q_1\, q_2 \ldots$ is called a *run* of $\mathcal{A}$ for $a_1 a_2 a_3 \ldots \in \Sigma^\omega$, if for every $i \in \mathbb{N}_{>0}$: $q_i \in \delta(q_{i-1}, a_i)$. A run is called *accepting* if it satisfies the acceptance condition Acc. The accepted language of $\mathcal{A}$ contains exactly those words $w \in \Sigma^\omega$ for which there exists an accepting run.

Definition 2.12. *A* deterministic Rabin automaton (DRA) *is an* ω*-automaton, where* Acc *is given as a set of tuples* $(L, K) \in 2^Q \times 2^Q$ *and a run is accepting, if there exists a pair* (L, K) *for which all states of* L *are visited only finitely often and at least one state of* K *is visited infinitely often. This condition is also referred to as* Rabin condition*.*

Definition 2.13. *A* Streett automaton *is an* ω*-automaton, where* Acc *is given as a set of tuples* $(E, F) \in 2^Q \times 2^Q$ *and a run is accepting if for every tuple* (E, F) *the following holds. If any state of* F *is visited infinitely often, then also a state of* E *is visited infinitely often. This condition is also referred to as* Streett condition*.*

In fact deterministic Rabin automata and Streett automata are equivalent with respect to their accepted languages, i.e., if a language is accepted by a deterministic Rabin automaton, there exists a Streett automaton accepting this language and vice-versa [26].

Further, the class of linear-time properties encodable using a deterministic Rabin or Streett automaton coincides with the well-studied class of *omega-regular* properties. One nice property of omega-regular properties is the following: Given an omega-regular property ϕ encoded by a deterministic Rabin automaton $\mathcal{A}$ and a Markov chain $\mathcal{M}$, model-checking ϕ for $\mathcal{M}$ can be done in time polynomial in both the size of $\mathcal{A}$ and $\mathcal{M}$. The main idea is to construct the product Markov chain $\mathcal{M} \otimes \mathcal{A}$ and then compute the probability of reaching any BSCC of $\mathcal{M} \otimes \mathcal{A}$ which satisfies the Rabin condition of $\mathcal{A}$ [1, Chapter 10.3].

Definition 2.14. *For a Markov chain* $\mathcal{M} = (S, P, \iota, AP, L)$ *and a deterministic Rabin automaton* $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, Acc)$ *, where* δ *is a total transition function and* $Acc = \{(L_1, K_1), \ldots, (L_k, K_k)\}$*, the product Markov chain* $\mathcal{M} \otimes \mathcal{A}$ *is defined as*

$$\mathcal{M} \otimes \mathcal{A} = (S \times Q, P', \iota', \{L_1, \ldots, L_k, K_1, \ldots, K_k\}, L'),$$

where $\iota' = \langle \iota, \delta(q_0, L(\iota)) \rangle$*,* $L'(\langle s, q \rangle) = \{L_i :\ q \in L_i\} \cup \{K_i :\ q \in K_i\}$ *and*

$$P'(\langle s, q \rangle, \langle s', q' \rangle) = \begin{cases} P(s, s') & \text{if } q' = \delta(q, L(s')) \\ 0 & \text{otherwise} \end{cases}$$

LTL-style notations

In the previous section we introduced weighted and energy-utility Markov chains, which extend Markov chains by a weight function or two reward functions, respectively. For these special Markov chains another interesting aspect of their behaviour is the accumulated weight or ratio. However, linear-time properties are restricted to the behaviour of Markov chains with respect to their labelling functions. To specify this kind of behaviour we will use notation inspired by linear temporal logic (LTL). LTL can be used to specify linear-time properties. However, we forego an introduction of LTL, as full LTL is beyond the scope of this thesis.

Instead, for a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ we only consider the following events. Given $z \in \mathbb{Z}$, $\bowtie \in \{<, \leq, =, \geq, >\}$ and a linear-time property ϕ, the events $\Box(\mathsf{weight} \bowtie z) \wedge \phi$ and $\Diamond(\mathsf{weight} \bowtie z) \wedge \phi$ are given by

$$\begin{aligned}
\pi \models \Box(\mathsf{weight} \bowtie z) \wedge \phi \qquad & \text{iff} \quad \mathsf{weight}(\pi[\dots n]) \bowtie z \text{ for all } n \in \mathbb{N}_{>0} \\
& \phantom{\text{iff}} \quad \text{and } \mathit{trace}(\pi) \in \phi, \\
\pi \models \Diamond(\mathsf{weight} \bowtie z) \wedge \phi \qquad & \text{iff} \quad \mathsf{weight}(\pi[\dots n]) \bowtie z \text{ for some } n \in \mathbb{N}_{>0} \\
& \phantom{\text{iff}} \quad \text{and } \mathit{trace}(\pi) \in \phi,
\end{aligned}$$

where $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$. Intuitively, π models $\Box(\mathsf{weight} \bowtie z) \wedge \phi$ if π satisfies ϕ and its accumulated weight *always* satisfies $\bowtie z$. To model $\Diamond(\mathsf{weight} \bowtie z)$ the accumulated weight must *eventually* satisfy $\bowtie z$. Analogously for the modalities $\Diamond\Box$ and $\Box\Diamond$,

$$\begin{aligned}
\pi \models \Diamond\Box(\mathsf{weight} \bowtie z) \wedge \phi \qquad & \text{iff} \quad \text{there exists } k \in \mathbb{N}_{>0} \text{ s.t. for all } n \in \mathbb{N}_{\geq k} \\
& \phantom{\text{iff}} \quad \mathsf{weight}(\pi[\dots n]) \bowtie z \text{ and } \mathit{trace}(\pi) \in \phi, \\
\pi \models \Box\Diamond(\mathsf{weight} \bowtie z) \wedge \phi \qquad & \text{iff} \quad \text{there exist infinitely many } n \in \mathbb{N}_{>0} \text{ with} \\
& \phantom{\text{iff}} \quad \mathsf{weight}(\pi[\dots n]) \bowtie z \text{ and } \mathit{trace}(\pi) \in \phi.
\end{aligned}$$

Example 2.15. Consider the weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ depicted in Figure 2.2 where $AP = S$ and $L(s) = \{s\}$ for all states $s \in S$. Further, let ϕ_1 be the set of all traces over 2^{AP} which satisfy the following Rabin condition $(\{s_2\}, \{s_3\})$ and ϕ_2 be the set of all traces which satisfy the Streett condition $\{(\{s_1\}, \{s_3\}), (\{s_2\}, \{s_3\})\}$.

Let us now consider the path $\pi = \iota\, t_1\, s_1\, (s_3)^\omega$. We have:

$$\begin{aligned}
&\pi \models \Box(\mathsf{weight} > 3) \wedge \phi_1 \qquad && \pi \models \Diamond(\mathsf{weight} > 100) \wedge \phi_1 \\
&\pi \not\models \Box(\mathsf{weight} > 3) \wedge \phi_2 \qquad && \pi \not\models \Diamond\Box(\mathsf{weight} < 5)
\end{aligned}$$

Given an energy-utility Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{energy}, \mathsf{utility})$, a $q \in \mathbb{Q}$, $\bowtie \in \{<, \leq, =, \geq, >\}$ and a linear-time property ϕ, the events $\heartsuit(\mathsf{ratio} \bowtie q)$ for

$\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ are defined as in the weighted case, i.e., for $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$

$$\begin{array}{lll}
\pi \models \Box(\mathsf{ratio} \bowtie q) \wedge \phi & \text{iff} & \mathsf{ratio}(\pi[\ldots n]) \bowtie q \text{ for all } n \in \mathbb{N}_{>0} \\
& & \text{and } \mathit{trace}(\pi) \in \phi, \\
\pi \models \Diamond(\mathsf{ratio} \bowtie q) \wedge \phi & \text{iff} & \mathsf{ratio}(\pi[\ldots n]) \bowtie q \text{ for some } n \in \mathbb{N}_{>0} \\
& & \text{and } \mathit{trace}(\pi) \in \phi, \\
\pi \models \Diamond\Box(\mathsf{ratio} \bowtie q) \wedge \phi & \text{iff} & \text{there exists } k \in \mathbb{N}_{>0} \text{ s.t. for all } n \in \mathbb{N}_{\geq k} \\
& & \mathsf{ratio}(\pi[\ldots n]) \bowtie q \text{ and } \mathit{trace}(\pi) \in \phi, \\
\pi \models \Box\Diamond(\mathsf{ratio} \bowtie q) \wedge \phi & \text{iff} & \text{there exist infinitely many } n \in \mathbb{N}_{>0} \text{ with} \\
& & \mathsf{ratio}(\pi[\ldots n]) \bowtie q \text{ and } \mathit{trace}(\pi) \in \phi.
\end{array}$$

2.3 Random walks

Definition 2.16. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables $X_n \colon \Omega \to \mathbb{Z}$, $n \in \mathbb{N}$ is called* Ω-random walk *or simply* random walk*, if the sequence $(\Delta X_n)_{n\in\mathbb{N}}$ is independent and identically distributed (i.i.d.), where for every $n \in \mathbb{N}$ the random variable $\Delta X_n \colon \Omega \to \mathbb{Z}$ is defined by*

$$\Delta X_n = X_{n+1} - X_n.$$

A random walk is said to be finite *if the random variables X_n have a finite image.*

One prominent example for random walks is the *Gambler's ruin*:

Example 2.17. Consider a gambler with finite wealth of w dollars, who continuously bets on a coin toss. If the coin shows heads, the gambler gets one dollar, otherwise he has to pay one dollar. For the sake of simplicity let us assume that the coin is fair.

This situation can be modeled with the following random walk. For every $n \in \mathbb{N}$ the random variable X_n describes the gambler's fortune at time n, i.e., after the n-th toss. As the coin is fair, we have for every $n \in \mathbb{N}$

$$P(\Delta X_n = +1) = 0.5 \qquad P(\Delta X_n = -1) = 0.5.$$

This random walk can also be modeled by the weighted Markov chain depicted in Figure 2.3.

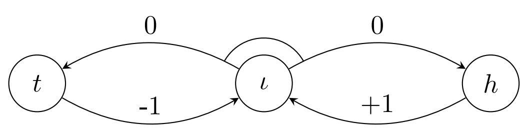

Figure 2.3: Weighted Markov chain modelling the Gambler's ruin

The strong law of large numbers is a fundamental theorem in probability theory. Applied to random walks it reads as follows.

Theorem 2.18. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $(Y_n)_{n\in\mathbb{N}}$ be a independent identically distributed sequence of random variables $Y_n \colon \Omega \to \mathbb{Z}$, $n \in \mathbb{N}$, such that $\mathbb{E}_{\mathcal{P}}(|Y_0|) < \infty$. Then for $\mathcal{P}$-almost all $\omega \in \Omega$, the sequence $(1/n+1) \cdot \sum_{k=0}^{n} Y_k(\omega)_{n\in\mathbb{N}}$ converges, and*

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot \sum_{k=0}^{n} Y_k(\omega) = \mathbb{E}_{\mathcal{P}}(Y_0).$$

One interesting property of random walks is whether it almost-surely returns to its initial value.

Definition 2.19. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. An Ω-random walk $(X_n)_{n\in\mathbb{N}}$ is called* recurrent*, if for every $z \in \mathbb{Z}$, the following implication holds: If there exists $n \in \mathbb{N}$, such that $P(X_n = z) > 0$, then for P-almost all $\omega \in \Omega$ there exist infinitely many $i \in \mathbb{N}$ such that $X_i(\omega) = z$.*

Example 2.20. Reconsider the random walk introduced in Example 2.17. This random walk is known to be recurrent. Let us discuss what recurrence means for the gambler's fortune. Notice that for every $n \in \mathbb{N}$ the probability that the coin shows tail n-times in a row is greater than zero. Thus, for every $z \in \mathbb{Z}_{<0}$ we have $P(X_{|z|+w} = z) > 0$, where w denotes the gambler's initial fortune. Thus, recurrence implies that the gambler goes bankrupt almost-surely, no matter how high his initial fortune was.

In this thesis, we will exploit the following two results for recurrent random-walks.

Theorem 2.21 ([21], Lemma 3.2, Theorem 8.1)**.** *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. An Ω-random walk $(X_n)_{n\in\mathbb{N}}$ is recurrent if for $\mathcal{P}$-almost all $\omega \in \Omega$, the sequence $(X_m(\omega)/n)_{n\in\mathbb{N}}$ converges and*

$$\lim_{n\to\mathbb{N}} \frac{X_n(\omega)}{n} = 0.$$

Theorem 2.22 ([21], Theorem 3.23)**.** *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Let $X = (X_n)_{n\in\mathbb{N}}$ be a recurrent Ω-random walk. The set*

$$\{z \in \mathbb{Z} \colon \text{ there exists } n \in \mathbb{N} \text{ such that } \mathcal{P}(X_n = z) > 0\}$$

constitutes a subgroup of the group $(\mathbb{Z}, +, 0)$. In particular, there exists $n \in \mathbb{N}$ such that $\mathcal{P}(X_n = 0) > 0$.

Notice that the authors in [21] study random walks with range $\mathbb{R}$. As we only consider random walks with range $\mathbb{Z}$, our definition of recurrence is less complex than the one given in [21] but the results are transferable.

3 Motivation and problem definition

In this chapter we will formally introduce the problems considered in this thesis. In order to motivate the presented problems, let us first consider an example which will serve as running example in the following chapters.

Example 3.1. Consider the pointed weighted Markov chain depicted in Figure 3.1. It models a system charging and draining a battery. During a transition with positive weight, the battery is charged, whereas negative weight implies that battery is discharged. The states ι, t_1, t_2, t_3 capture the behaviour in the system's initialization phase. The BSCCs $\mathcal{C}_1 = \{s_1, s_2, s_3\}$, $\mathcal{C}_2 = \{s_4, s_5, s_6\}$ and $\mathcal{C}_3 = \{s_7, s_8, s_9\}$ model different operation modes, which are reached dependent on the initialization phase. For the sake of simplicity this dependency is modeled by an uniformly distributed probability for the outgoing transitions of state t_3. Notice that the operation modes differ only in their charging efficiency. If the system reaches s_1, the probability that the system returns to s_1 with the same or a higher battery level is $0.75 = 1 - P(s_1, s_2) \cdot P(s_2, s_3) \cdot P(s_3, s_1)$. In contrast, for state s_7 the probability is only 0.3125.

When analyzing and evaluating the system introduced in Example 3.1, it might be interesting to know the probability that the system eventually runs out of battery when started with a given initial battery level l. Formally,

$$1 - Pr_{\iota}^{\mathcal{M}}(\Box(\mathsf{weight} > -l)).$$

However, maybe there are additional requirements to be met. Assume for example that the only states, where the system actually gets work done, are s_2, s_5 and s_8. We thus additionally require that at least one productive state is visited infinitely often. This condition can be encoded using the following Rabin-condition $\{(\{\}, \{s_2, s_5, s_8\})\}$ and thus can be represented by an omega-regular property ϕ. Further we might not be interested in the exact probability that the system does not run out of battery while satisfying ϕ, but only whether the probability is greater than a given threshold p, i.e.

$$Pr_{\iota}^{\mathcal{M}}(\Box(\mathsf{weight} > -l) \wedge \phi) > p.$$

This quantitative weight problem is known to be decidable in EXPSPACE [4]. *Qualitative weight problems* are a special case of quantitative problems, where we only ask if the probability is greater than 0 or equal to 1.

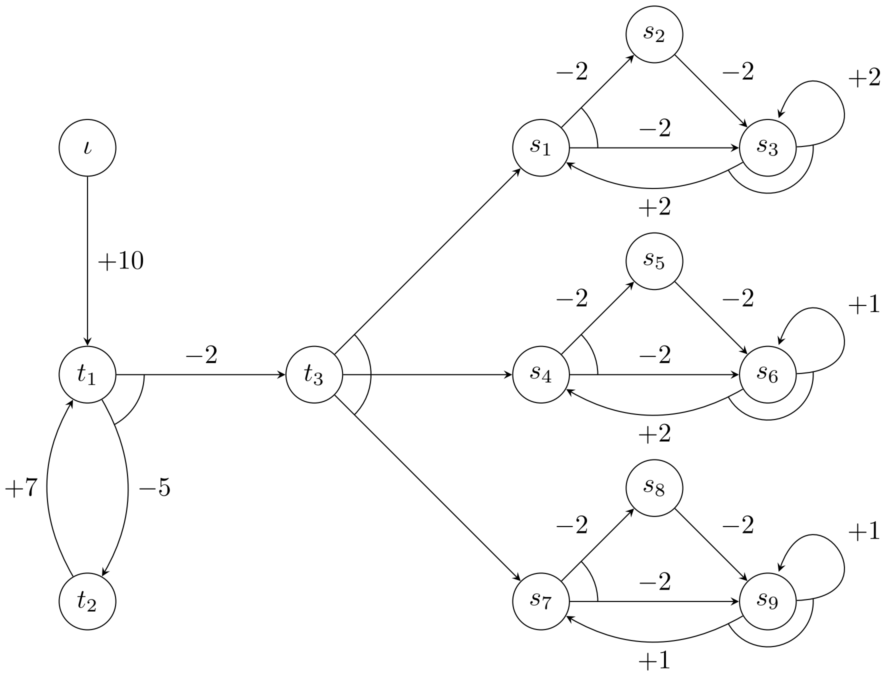

Figure 3.1: Markov Chain (choices are assumed to be uniformly distributed)

Definition 3.2. *Let* $\mathcal{M} = (S, \iota, P, AP, L, \mathsf{weight})$ *be a weighted Markov chain,* $z \in \mathbb{Z}$, $\heartsuit$ *one of the four temporal modalities* $\Box$, $\Diamond$, $\Box\Diamond$, $\Diamond\Box$, *and* ϕ *an omega-regular property over* AP*. We define the following* qualitative $\heartsuit$-weight problems*:*

positive $\heartsuit$*-weight problem*: *does* $Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) > 0$ *hold?*

almost-sure $\heartsuit$*-weight problem*: *does* $Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) = 1$ *hold?*

Analogously, let us define qualitative ratio problems for energy-utility Markov chains.

Definition 3.3. *Let* $\mathcal{M} = (S, \iota, P, AP, L, \mathsf{energy}, \mathsf{utility})$ *be an energy-utility Markov chain,* $q \in \mathbb{Q}$, $\heartsuit$, ϕ *as in Definition 3.2 and* $\mathsf{ratio} = \mathsf{energy}/\mathsf{utility}$ *as usual. We define the following* qualitative $\heartsuit$-ratio problems*:*

positive $\heartsuit$*-ratio problem*: *does* $Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{ratio} > q) \wedge \phi) > 0$ *hold?*

almost-sure $\heartsuit$*-ratio problem*: *does* $Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{ratio} > q) \wedge \phi) = 1$ *hold?*

By solving the decision problems defined above, we can now decide whether or not the system may run out of battery. However, assume that we want to know the lowest initial battery level which ensures that the system does not run out of battery. Then the above problems are just of limited use to answer our questions, as they require that we guess an initial battery level of interest. Instead, what we are looking for are quantiles.

Definition 3.4. *Let $\mathcal{M} = (S, \iota, P, AP, L, \mathsf{weight})$ be a weighted Markov chain, ϕ and $\heartsuit$ as in Definition 3.2. We define the following* qualitative $\heartsuit$-weight quantiles*:*

positive $\heartsuit$-weight quantile:

$$\mathsf{Qu}_{\phi}^{>0}[\heartsuit\mathsf{weight}] = \sup\{z \in \mathbb{Z}\colon Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) > 0\}$$

almost-sure $\heartsuit$-weight quantile:

$$\mathsf{Qu}_{\phi}^{=1}[\heartsuit\mathsf{weight}] = \sup\{z \in \mathbb{Z}\colon Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) = 1\}$$

Definition 3.5. *Let $\mathcal{M} = (S, \iota, P, AP, L, \mathsf{energy}, \mathsf{utility})$ be an energy-utility Markov chain, $\heartsuit$, ϕ, as in Definition 3.2 and* $\mathsf{ratio} = \mathsf{energy}/\mathsf{utility}$ *as before. We define the following* qualitative $\heartsuit$-ratio quantiles*:*

positive $\heartsuit$-ratio quantile:

$$\mathsf{Qu}_{\phi}^{>0}[\heartsuit\mathsf{ratio}] = \sup\{q \in \mathbb{Q}\colon Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{ratio} > z) \wedge \phi) > 0\}$$

almost-sure $\heartsuit$-ratio quantile:

$$\mathsf{Qu}_{\phi}^{=1}[\heartsuit\mathsf{ratio}] = \sup\{q \in \mathbb{Q}\colon Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{ratio} > z) \wedge \phi) = 1\}$$

If $\phi = (2^{AP})^{\omega}$ we write $\mathsf{Qu}^*[\heartsuit\mathsf{weight}]$ and $\mathsf{Qu}^*[\heartsuit\mathsf{ratio}]$ instead of $\mathsf{Qu}_{\phi}^*[\heartsuit\mathsf{weight}]$ and $\mathsf{Qu}_{\phi}^*[\heartsuit\mathsf{ratio}]$ respectively, where $* \in \{>0, =1\}$.

Beside best- and worst-case analysis, these quantiles could also be used to compare two systems.

Example 3.6. Reconsider Example 3.1. Beside the battery level, we might be interested in the systems energy efficiency. To this end, let us consider the energy-utility Markov chain depicted in Figure 3.2, where each transition is assigned two natural numbers, the first stands for the gained utility and the second for the consumed energy of this transition. Notice, the model implies that the system's efficiency does not depend on the operation mode reached. However, let us assume that there exist three variants v_1, v_2 and v_3 of our system, which differ both in their price and the energy consumption for the transition from t_1 to c_1 ie.

$$\mathsf{energy}_{v_1}(t_1, c_1) = 2 \qquad \mathsf{energy}_{v_2}(t_1, c_1) = 4 \qquad \mathsf{energy}_{c_3}(t_1, c_1) = 8.$$

It is now left to the engineer to decide which variant to buy.

As mentioned in Chapter 1, one way to evaluate the trade-off of using a more expensive variant would be to compute and compare qualitative ratio quantiles for the alternatives.

In this thesis we will show that all qualitative problems and qualitative quantiles stated above are solvable in polynomial time. Chapter 4 is dedicated to prove decidability in polynomial time under the assumption that the side constraint ϕ is omitted, i.e., $\phi = (2^{AP})^{\omega}$. In Chapter 5 we show that under this assumption qualitative quantiles are computable in polynomial time In Chapter 6 we then generalize the results for omega-regular properties encoded by a deterministic Rabin or Street automaton $\mathcal{A}$ and show that both, qualitative decision problems and qualitative quantiles, are solvable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$.

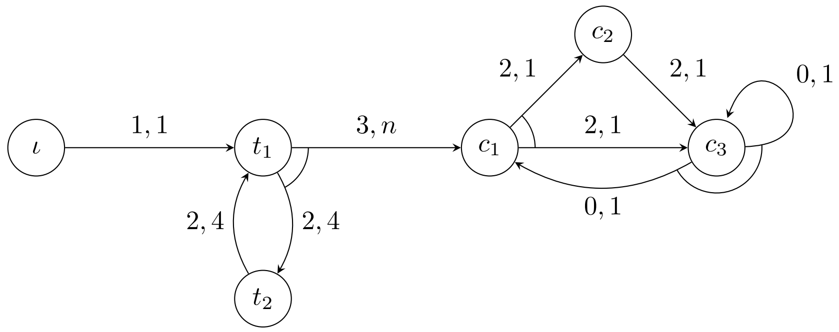

Figure 3.2: Finite energy-utility Markov chain (transitions are labeled with tuples (utility,energy) and choices are assumed to be uniformly distributed)

4 Towards a polynomial decision procedure

In this chapter, we show that both, qualitative weight problems and qualitative ratio problems, are decidable in polynomial time. As mentioned before, we first assume that the linear-time side constraint appearing in the qualitative decision problems is simply $\phi = (2^{AP})^\omega$ and generalize the results for omega-regular properties in Chapter 6. As under this assumption AP and L can be omitted, we will abuse notation and denote weighted Markov chains by the tuple $(S, P, \iota, \mathsf{weight})$ and energy-utility Markov chains by $(S, P, \iota, \mathsf{energy}, \mathsf{utility})$.

It is well-known that qualitative ratio problems can be reduced to qualitative weight problems in polynomial time [6, 4]. Using the following lemma, obtained results for qualitative weight problems can be directly transferred to qualitative ratio problems, and hence, for the remainder of this chapter, we restrict ourselves to weighted decision problems.

Lemma 4.1. *Let* $\mathcal{M} = (S, P, \iota, \mathsf{energy}, \mathsf{utility})$ *be an energy-utility Markov chain,* $\heartsuit \in \{\Box, \Diamond, \Box\Diamond, \Diamond\Box\}$ *and* $n_1, n_2 \in \mathbb{N}_{>0}$. *Let* $\mathsf{weight}_{(n_1/n_2)} = n_2 \cdot \mathsf{utility} - n_1 \cdot \mathsf{energy}$. *For every path* $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$

$$\pi \models \heartsuit\left(\frac{\mathsf{utility}}{\mathsf{energy}} > \frac{n_1}{n_2}\right) \qquad \textit{iff} \qquad \pi \models \heartsuit(\mathsf{weight}_{n_1/n_2} > 0).$$

Using similar techniques as Brázdil et al. in [10], we will show that in order to decide qualitative $\heartsuit$-weight problems, it suffices to analyze both the BSCCs regarding to their expected weight and the possibilities to reach these BSCCs.

The BSCC analysis is based on a transformation to random walks, which is introduced in Section 4.1 and exploited in Section 4.2 to characterize BSCCs with respect to qualitative $\heartsuit$-weight objectives. In Section 4.3 we define some basic functions on the underlying graph of a Markov chain, which in Section 4.4 allows us to lift the results for BSCCs to arbitrary Markov chains and provide characterizations for qualitative $\heartsuit$-weight problems. In Section 4.5 we show that all ingredients of these characterizations can be computed in polynomial time and thus the qualitative $\heartsuit$-weight problems are decidable in polynomial time.

For an intuition of the role of expected weights for BSCCs consider Example 3.1. The system's behaviour heavily depends on the chosen operation mode, i.e., the reached BSCC. Intuitively, given an initial battery level, the probability of running out of battery is greater in any state of BSCC $\mathcal{C}_3 = \{s_7, s_8, s_9\}$ than in a state of $\mathcal{C}_1 = \{s_1, s_2, s_3\}$. Notice that $\mathcal{C}_1$ and $\mathcal{C}_3$ differ only in their weights. Thus, their steady-state distribution is identical but they have a different expected weight.

In the following we will show that when ever the system enters a BSCC with negative expected weight, it will run out of battery, no matter how high the initial battery level was. The same applies for BSCCs with zero expected weight which contain negative cycles. For all other BSCCs there exists an initial battery level, such that with positive probability the system will not run out of battery. Thus, the system will almost surely run out of battery if enters operation mode $\mathcal{C}_2 = \{s_4, s_5, s_6\}$ or $\mathcal{C}_3$, as $\mathbb{E}_{\mathcal{C}_2} = 0$ and $\mathbb{E}_{\mathcal{C}_3} < 0$. In contrast, we will show that with positive probability the system will not run out of battery, if it enters operation mode $\mathcal{C}_1$ with a battery level of at least 2.

Before we dive into the transformation of a BSCC into a random walk which we will use to show that qualitative $\heartsuit$-weight problems are decidable in polynomial time, let us see why it suffices to restrict ourselves to the modalities $\Box$ and $\Diamond\Box$. In the remainder of this chapter let $\mathcal{M} = (S, \iota, P, \mathsf{weight})$ be a weighted Markov chain and $z \in \mathbb{Z}$ an integer.

Lemma 4.2.

$$Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{weight} > z)) = 1 - Pr_\iota^{\mathcal{M}}(\Box(-\mathsf{weight} > -(z+1)))$$
$$Pr_\iota^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} > z)) = 1 - Pr_\iota^{\mathcal{M}}(\Diamond\Box(-\mathsf{weight} > -(z+1)))$$

Proof. We here give the formal arguments for $\Diamond$, the case $\Box\Diamond$ is analogous. Using $P(A) = 1 - P(\neg A)$ and $a < b \Leftrightarrow -a > -b$ we have

$$\begin{aligned} Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{weight} > z)) &= 1 - Pr_\iota^{\mathcal{M}}(\neg(\Diamond(\mathsf{weight} > z))) \\ &= 1 - Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} \leq z)) \\ &= 1 - Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} < z+1)) \\ &= 1 - Pr_\iota^{\mathcal{M}}(\Box(-\mathsf{weight} > -(z+1))) \end{aligned}$$

$\Box$

4.1 From BSCCs to random walks

The aim of this section is to state and analyze a transformation of a BSCC $\mathcal{C}$ into a random walk. This random walk will then be used in Section 4.2 to classify BSCCs with respect to their expected weight.

The transformation is based on the fact that within a BSCCs almost-surely every state is visited infinitely often. For the remainder of this section, let us fix a BSCC $\mathcal{C}$ of $\mathcal{M}$ and a state $s \in \mathcal{C}$. Hence, almost all paths contained in $\mathsf{InfPaths}_s^{\mathcal{M}}$ can be seen as a concatenation of infinitely many cycles starting in s. This motivates the next definition.

Definition 4.3. *For every $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ and $n \in \mathbb{N}$, we introduce the path $\pi[\ldots \uparrow n]$ of $\mathcal{M}$ as follows: $\pi[\ldots \uparrow n]$ is defined to be the minimal prefix of π, that visits s at least $n+1$ times. If such prefix does not exists, let $\pi[\ldots \uparrow n] = \pi$.*

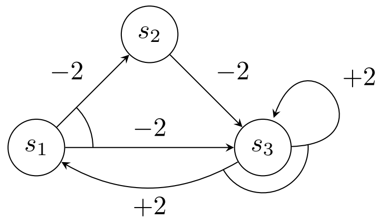

Figure 4.1: weighted BSCC (isomorphic to BSCC $\{s_1, s_2, s_3\}$ of Figure 3.1)

Example 4.4. Consider the BSCC $\mathcal{C}$ depicted in Figure 4.1. Notice that it constitutes a single BSCC. Let $\pi \in \mathsf{InfPaths}^{\mathcal{C}}$ such that $\pi \in \mathit{Cyl}(s_1 s_2 s_3 s_3 s_1 s_3 s_1 s_2 s_3 s_1 s_3)$. Then,

$$\begin{aligned} \pi[\ldots \uparrow 0] &= s_1 \\ \pi[\ldots \uparrow 1] &= s_1 s_2 s_3 s_3 s_1 \\ \pi[\ldots \uparrow 2] &= s_1 s_2 s_3 s_3 s_1 s_3 s_1. \end{aligned}$$

We now use Definition 4.3 to define two random walks, which both are based on the idea to consider every cycle ϑ from s to s as one "step" of the random walk. Whereas the first assigns the step the length of ϑ the second uses the accumulated weight of ϑ.

Definition 4.5. *We define the* $\mathsf{InfPaths}^{\mathcal{M}}_s$*-random walks* $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ *and* $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ *as follows: for every* $n \in \mathbb{N}$ *and* $\pi \in \mathsf{InfPaths}^{\mathcal{M}}_s$ *let*

$$\begin{aligned} \mathsf{Len}_n(\pi) &= |\pi[\ldots \uparrow n]|, \\ \mathsf{Wgt}_n(\pi) &= \mathsf{weight}(\pi[\ldots \uparrow n]), \end{aligned}$$

if $|\pi[\ldots \uparrow n]| < \infty$*, and* $\mathsf{Wgt}_n(\pi) = \mathsf{Len}_n(\pi) = 0$ *otherwise.*

Notice, since $\mathsf{Len}_0 = 0$ and $\mathsf{Wgt}_0 = 0$, for every $n \in \mathbb{N}$

$$\mathsf{Len}_{n+1} = \sum_{k=0}^{n} \Delta\mathsf{Len}_k \quad \text{and} \quad \mathsf{Wgt}_{n+1} = \sum_{k=0}^{n} \Delta\mathsf{Wgt}_k.$$

Example 4.6. Consider the Markov chain $\mathcal{C}$ in Figure 4.1, which constitutes a BSCC and let $s = s_3$. Assume $\pi \in \mathsf{InfPaths}^{\mathcal{C}}_s$ and $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{C}}_s$ such that for some $n \in \mathbb{N}$, $\pi[\ldots \uparrow n{+}1] = \pi[\ldots \uparrow n] \diamond \hat{\pi}$. Thus $\hat{\pi} \in \{s_3 s_1 s_3,\ s_3 s_1 s_2 s_3,\ s_3 s_3\}$ and both $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ and $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ are finite random walks where for every $i \in \mathbb{N}$ with:

$$\begin{aligned} P(\Delta\mathsf{Len}_i = 1) &= 0.5 & P(\Delta\mathsf{Wgt}_i = -2) &= 0.25 \\ P(\Delta\mathsf{Len}_i = 2) &= 0.25 & P(\Delta\mathsf{Wgt}_i = 0) &= 0.25 \\ P(\Delta\mathsf{Len}_i = 3) &= 0.25 & P(\Delta\mathsf{Wgt}_i = +2) &= 0.5 \end{aligned}$$

However $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ and $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ do not have to be finite. Assume $s = s_1$ and $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{C}}_s$. Then $\pi[\ldots \uparrow n+1] = \pi[\ldots \uparrow n] \diamond \hat{\pi}$ for some $\pi \in \mathsf{InfPaths}^{\mathcal{C}}_s$

implies $\hat{\pi} \in \{s_1 s_2 s_3 (s_3)^* s_1,\ s_1 s_3 (s_3)^* s_1\}$. Then $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ and $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ are infinite random walks where for every $i \in \mathbb{N}$

$$P(\Delta\mathsf{Len}_i = 2) = 0.25 \qquad P(\Delta\mathsf{Len}_i = m_1) = 0.25 \cdot 0.5^{m_1-3} + 0.25 \cdot 0.5^{m_1-2}$$
$$P(\Delta\mathsf{Wgt}_i = -2) = 0.25 \qquad P(\Delta\mathsf{Wgt}_i = 2m_2) = 0.25 \cdot 0.5^{m_2+1} + 0.25 \cdot 0.5^{m_2}$$

where $m_1 \in \mathbb{N}_{\geq 3}$ and $m_2 \in \mathbb{N}$.

Notice that Definition 4.5 implicitly assumes that both $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ and $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ indeed constitute random walks, which has to be proven first.

Lemma 4.7. *Both* $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ *and* $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ *constitute random walks.*

Proof. It suffices to show that $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ constitutes a random walk, as $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ is a corner case of $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ where $\mathsf{weight}(s, s') = 1$ for all $s, s' \in S$.

We first prove that $(\Delta\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ is identically distributed, by calculating the probability $Pr_s^{\mathcal{M}}(\Delta\mathsf{Wgt}_n = w)$ for a given $w \in \mathbb{Z}$ and showing that this value does not depend on n. Let $n \in \mathbb{N}$, $w \in \mathbb{Z}$, and $A, B \subseteq S^*$ be given as follows,

$$\begin{aligned} A &= (\{s\} \cdot (S \setminus \{s\})^*)^n \cdot \{s\}, \\ B &= \{\hat{\pi} \in \{s\} \cdot (S \setminus \{s\})^* \cdot \{s\} \colon \mathsf{weight}(\hat{\pi}) = w\}. \end{aligned}$$

It holds $(\Delta\mathsf{Wgt}_n = w)^{-1} = A \diamond B \diamond (\{s\} \cdot S^\omega) \cap \mathsf{InfPaths}_s^{\mathcal{M}}$. For a finite path $\hat{\pi} \in \mathsf{FinPaths}$, let $P(\hat{\pi}) = \prod_{i=1}^{|\hat{\pi}|} P(s_{i-1}, s_i)$. Since s is a state of a BSCC, $Pr_s^{\mathcal{M}}$-almost all paths visit s infinitely often, which implies $\sum_{\hat{\pi}_A \in A} P(\hat{\pi}_A) = 1$. Hence,

$$Pr_s^{\mathcal{M}}(\Delta\mathsf{Wgt}_n = w) = \sum_{\hat{\pi}_B \in B} \left(P(\hat{\pi}_B) \sum_{\hat{\pi}_A \in A} P(\hat{\pi}_A) \right) = \sum_{\hat{\pi}_B \in B} P(\hat{\pi}_B)$$

which, as the right-hand side does not depend on n, yields that $(\Delta\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ is identically distributed.

It remains to show that $(\Delta\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ is independent. Let $k \in \mathbb{N}$, $\{z_0, \ldots, z_k\} \subset \mathbb{Z}$, and $\{i_0, \ldots, i_k\} \subset \mathbb{N}$ be such that $i_0 < \ldots < i_k$. We define $F \in \mathsf{InfPaths}_s^{\mathcal{M}}$ by

$$F = \{\pi \in \mathsf{InfPaths}_s^{\mathcal{M}} \colon \Delta\mathsf{Wgt}_{i_j}(\pi) = z_j \quad \forall 0 \leq j \leq k\}.$$

In the following we will show $Pr_s^{\mathcal{M}}(F) = \prod_{j=0}^{k} Pr_s^{\mathcal{M}}(\Delta\mathsf{Wgt}_{i_j} = z_j)$ which implies the independence of $(\Delta\mathsf{Wgt}_n)_{n\in\mathbb{N}}$.

To do so, for every path $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ visiting s infinitely often, let the sequence $(\hat{\pi}_n)_{n\in\mathbb{N}}$ be given by $s\hat{\pi}_0 s\hat{\pi}_1 s\hat{\pi}_2 s \ldots = \pi$, where $\hat{\pi}_n \in (S \setminus \{s\})^*, n \in \mathbb{N}$. Further we define $F' \subseteq \mathsf{FinPaths}_s^{\mathcal{M}}$ as the set containing all finite paths $\hat{\pi}$ of $\mathcal{M}$ such that there exists $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ with $\hat{\pi} = \pi[\ldots \uparrow i_k]$ and for all $0 \leq j \leq k$, $\mathsf{weight}(s\hat{\pi}_{i_j} s) = z_j$. The set F is $Pr_s^{\mathcal{M}}$-almost surely equivalent to $\bigcup_{\hat{\pi} \in F'} \mathsf{Cyl}(\hat{\pi})$. Thus,

$$\begin{aligned} Pr_s^{\mathcal{M}}(F) &= \sum_{\hat{\pi} \in F'} P(\hat{\pi}) = \sum_{\hat{\pi} \in F'} P(s\hat{\pi}_{i_0} s) \cdot \ldots \cdot P(s\hat{\pi}_{i_k} s) \\ &= \left(\sum_{\hat{\pi}'_0 \in M_0} P(s\hat{\pi}'_0 s) \right) \cdot \ldots \cdot \left(\sum_{\hat{\pi}'_k \in M_k} P(s\hat{\pi}'_k s) \right) \end{aligned}$$

where for all $0 \leq j \leq k$ the set M_k contains all $\hat{\pi}'_j \in (S \setminus s)^*$ satisfying $\mathsf{weight}(s\hat{\pi}'_j s) = z_{i_j}$. As $\left(\sum_{\hat{\pi}'_j \in M_j} P(s\hat{\pi}'_j s)\right) = Pr_s^{\mathcal{M}}(\Delta\mathsf{Wgt}_{i_j} = z_{i_j})$ this completes the proof. □

Let us now analyze the random walks $(\mathsf{Len}_n)_{n\in\mathbb{N}}$ and $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$. One common prerequisite for stochastic lemmata is a bounded expected value. Using the finiteness of $\mathcal{C}$ we achieve this bound for $\mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len})$.

Lemma 4.8. *We have*

$$0 < \mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len}_0) = \mathbb{E}_s^{\mathcal{M}}(|\Delta\mathsf{Len}_0|) < +\infty.$$

Proof. Clearly, $0 < \mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len}_0) = \mathbb{E}_s^{\mathcal{M}}(|\Delta\mathsf{Len}_0|)$ as $\Delta\mathsf{Len}_0 : \mathsf{InfPaths}_s^{\mathcal{M}} \to \mathbb{N}$ and

$$Pr_s^{\mathcal{M}}(\Delta\mathsf{Len}_0 = 0) = Pr_s^{\mathcal{M}}(\mathsf{Len}_1 = 0) = 0.$$

The second inequality directly follows from basic results of Markov chain theory. Define the random variable $\tau : \mathsf{InfPaths}_s^{\mathcal{M}} \to \mathbb{N} \cup \{\infty\}$ for every $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ by $\tau(\pi) = \inf\{n \in \mathbb{N}_{>0} \colon \pi[n] = s\}$. Notice that $\mathbb{E}_s^{\mathcal{M}}(\tau)$ denotes the expected return time to state s. As s is contained in a finite BSCC $\mathcal{C}$, the expected return time to state s is finite, i.e., $\mathbb{E}_s^{\mathcal{M}}(\tau) < +\infty$ [16, Proposition 78 in Chapter 1]. As for $Pr_s^{\mathcal{M}}$-almost all infinite paths $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ we have $\Delta\mathsf{Len}_0(\pi) = \mathsf{Len}_1(\pi) = \tau(\pi)$, this yields the claim.

□

Lemma 4.9. *For $Pr_s^{\mathcal{M}}$-almost all $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$, the sequence $(\mathsf{Wgt}_n(\pi)/n)_{n\in\mathbb{N}}$ converges, and*

$$\lim_{n\to\infty} \frac{\mathsf{Wgt}_n(\pi)}{n} = \mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len}_0) \cdot \mathbb{E}_{\mathcal{C}}(\mathsf{weight})$$

Proof. Lemma 4.8 and the fact that $(\Delta\mathsf{Len}_n)_{n\in\mathbb{N}}$ is independent and identically distributed allow the application of Theorem 2.18, which yields for $Pr_s^{\mathcal{M}}$-almost all paths $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$

$$\mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len}_0) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Delta\mathsf{Len}_k(\pi) = \lim_{n\to\infty} \frac{\mathsf{Len}_{n+1}(\pi)}{n+1}. \tag{4.1}$$

Let us argue for $Pr_s^{\mathcal{M}}$-almost all infinite path $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$. Remember that for every $t, t' \in \mathcal{C}$, we have

$$\mathbb{S}_{\mathcal{C}}(t) \cdot P(t,t') = \lim_{n\to\infty} \frac{1}{n} \cdot |\{k \in \{0,\ldots,n\} \colon \pi[k] = t,\, \pi[k+1] = t'\}|.$$

Furthermore, $(\mathsf{Wgt}_n(\pi)/\mathsf{Len}_n(\pi))_{n\in\mathbb{N}}$ is a subsequence of $(\mathsf{weight}(\pi[\ldots n])/n)_{n\in\mathbb{N}}$. Hence,

$$\begin{aligned}\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) &= \sum_{t,t'\in\mathcal{C}} \mathsf{weight}(t) \cdot \mathbb{S}_{\mathcal{C}}(t) \cdot P(t,t') \\ &= \lim_{n\to\infty} \frac{\mathsf{weight}(\pi[\ldots n])}{n} = \lim_{n\to\infty} \frac{\mathsf{Wgt}_n(\pi)}{\mathsf{Len}_n(\pi)} \end{aligned} \tag{4.2}$$

Equation (4.1) and Equation (4.2) yield the claim as

$$\begin{aligned}\mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len}_0)\cdot\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) &= \lim_{n\to\infty}\frac{\mathsf{Len}_{n+1}(\pi)}{n+1}\cdot\lim_{n\to\infty}\frac{\mathsf{Wgt}_n(\pi)}{\mathsf{Len}_n(\pi)}\\ &= \lim_{n\to\infty}\frac{\mathsf{Len}_n(\pi)}{n}\cdot\lim_{n\to\infty}\frac{\mathsf{Wgt}_n(\pi)}{\mathsf{Len}_n(\pi)} = \lim_{n\to\infty}\frac{\mathsf{Wgt}_n(\pi)}{n}.\end{aligned}$$

□

4.2 Classifying BSCCs

Given a BSCC $\mathcal{C}$, exactly one of the following statements holds:

(a) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$.

(b) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ contains no negative cycle.

(c) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does contain a negative cycle.

(d) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$.

In this section we investigate these four classes of BSCCs separately. To this end, let us fix a BSCC $\mathcal{C}$ and a state $s \in \mathcal{C}$. Using the random walk defined in the previous section we state and prove a characterization for $z \in \mathbb{Z}$ and $\heartsuit \in \{\Box, \Diamond\Box\}$ such that

$$Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z)) > 0.$$

Further we show that if $\mathcal{C}$ satisfies either (c) and (d), we have for every $z \in \mathbb{Z}$

$$Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = 0.$$

Thus, if the system introduced in Example 3.1 enters operation mode $\mathcal{C}_2 = \{s_4,\, s_5,\, s_6\}$ or $\mathcal{C}_3 = \{s_7,\, s_8,\, s_9\}$ the system will almost-surely run out of battery, no matter how high the intial battery level.

Let us first treat case (d). Notice that all paths of the following event E_z, $z \in \mathbb{Z}$, neither satisfy $\Box(\mathsf{weight} > z)$ nor $\Diamond\Box(\mathsf{weight} > z)$:

$$E_z = \{\pi \in \mathsf{InfPaths}_s^{\mathcal{M}} :\ \mathsf{weight}(\pi[\ldots n]) \le z \text{ for infinitely many } n \in \mathbb{N}\}.$$

In the following we show that $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$ implies that there does not exist a lower bound for the accumulated weight, i.e., for every $z \in \mathbb{Z}$ the event E_z has probability 1.

Lemma 4.10. *Assuming* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$ *for* $Pr_s^{\mathcal{M}}$*-almost all infinite paths* π *of* $\mathcal{M}$

$$\inf_{n\in\mathbb{N}} \mathsf{weight}(\pi[\ldots n]) = \inf_{n\in\mathbb{N}} \mathsf{Wgt}_n(\pi) = -\infty.$$

Proof. The claim is a direct consequence of Lemma 4.9. We argue for $Pr_s^{\mathcal{M}}$-almost all paths π of $\mathcal{M}$. As $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$ and $\mathbb{E}_s^{\mathcal{M}}(\Delta\mathsf{Len}_0) > 0$ we have

$$\lim_{n\to\infty} \frac{\mathsf{Wgt}_n(\pi)}{n} < 0. \tag{4.3}$$

Hence $\inf_{n\in\mathbb{N}} \mathsf{Wgt}_n(\pi)$ is negative. Furthermore, $\inf_{n\in\mathbb{N}} \mathsf{Wgt}_n(\pi) \in \mathbb{Z}$ would imply that the sequence $(\mathsf{Wgt}_n(\pi)/n)$ tends to zero, which contradicts Equation (4.3). Thus, $\inf_{n\in\mathbb{N}} \mathsf{Wgt}_n(\pi) = -\infty$. Clearly, this implies $\inf_{n\to\infty} \mathsf{weight}(\pi[\ldots n]) = -\infty$, which yields the claim. □

Corollary 4.11. *Let* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$ *and* $\heartsuit \in \{\square, \lozenge\square\}$*. For every integer* $z \in \mathbb{Z}$ *holds*

$$Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z)) = 0.$$

Example 4.12. Consider again the Markov chain of Example 3.1 depicted in Figure 3.1. Corollary 4.11 yields that, no matter how high the system's battery level is at the time reaching s_7, the system will almost-surely eventually run out of battery.

Let us now consider the case $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$. Intuitively, the expected weight describes the average weight per transition. Thus, it seems natural to assume that almost all paths π in such BSCC infinitely often "hit" 0, i.e., there exist infinitely many $n \in \mathbb{N}$ such that $\mathsf{weight}(\pi[\ldots n]) = 0$. Formally, this claim follows directly from the results of the previous section.

Lemma 4.13. *If* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$*, then for* $Pr_s^{\mathcal{M}}$*-almost all paths* $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$*, the sequence* $(\mathsf{Wgt}_n(\pi)/n)_{n\in\mathbb{N}}$ *converges, and*

$$\lim_{n\to\infty} \frac{\mathsf{Wgt}_n(\pi)}{n} = 0.$$

In particular, the random walk $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ *is recurrent.*

Proof. The claim is a direct consequence of Theorem 2.21 and Lemma 4.9. □

As already mentioned at the beginning of this section, if $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$, we consider the cases where $\mathcal{C}$ does not contain negative cycles (case (b)) and where $\mathcal{C}$ contains negative cycles (case (c)) separately. Similar to $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$, for the latter we show that almost-all paths infinitely often accumulate arbitrary low weights. The proof is based on the following fact: If $\mathcal{C}$ contains a negative cycle, for every $z \in \mathbb{Z}$ there exists a cycle ϑ starting in s such that $\mathsf{weight}(\vartheta) < z$. Note that ϑ does not necessarily has to be a simple cycle. By Lemma 4.13 for almost-all paths $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ there exist infinitely many $n \in \mathbb{N}$ such that $\mathsf{weight}(\pi[\ldots n]) = 0$ and $\mathsf{last}(\pi[\ldots n]) = s$. Hence, by basic laws of probability theory there exist infinitely many $n \in \mathbb{N}$ such that $\pi = \pi[\ldots n] \diamond \vartheta \diamond \pi'$ for some $\pi' \in \mathsf{InfPaths}_s^{\mathcal{M}}$.

We prove the more general statement that almost-all paths infinitely often accumulate arbitrary high and low weights. To be able to apply the above idea, let us first prove the following statement.

Lemma 4.14. *Assume* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$. *Then,* $\mathcal{C}$ *contains a positive cycle, if and only if* $\mathcal{C}$ *contains a negative cycle.*

Proof. As by Lemma 4.13 $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ is recurrent, we can exploit Theorem 2.22.

Assume $\mathcal{C}$ contains a weight-positive cycle. This implies that there exists a (not necessarily simple) weight-positive cycle ϑ with $\mathsf{first}(\vartheta) = s$. For every $\pi \in \mathsf{Cyl}(\vartheta)$ we have $\mathsf{Wgt}_1(\pi) = \mathsf{weight}(\pi[\ldots \uparrow 1]) = \mathsf{weight}(\vartheta) > 0$. Thus,

$$Pr_s^{\mathcal{M}}(\mathsf{Wgt}_1 = \mathsf{weight}(\vartheta)) \geq Pr_s^{\mathcal{M}}(\mathsf{Cyl}(\vartheta)) > 0.$$

As $(\mathsf{Wgt}_n)_{n\in\mathbb{N}}$ is recurrent, Theorem 2.22 implies that there exists an $n \in \mathbb{N}$ such that $Pr_s^{\mathcal{M}}(\mathsf{Wgt}_n = -\mathsf{weight}(\vartheta)) > 0$. Thus, there exists a path $\pi' \in \mathsf{InfPaths}_s^{\mathcal{M}}$ such that $\mathsf{Wgt}_n(\pi') = \mathsf{weight}(\pi'[\ldots \uparrow n]) = -\mathsf{weight}(\vartheta)$. Hence $\pi'[\ldots \uparrow n]$ is a negative cycle. This proves that the existence of a positive cycle implies the existence of a negative cycle. The reverse implication can be proven analogously. $\square$

Lemma 4.15. *Assume that* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ *and that* $\mathcal{C}$ *contains a negative cycle. For every integer* $z \in \mathbb{Z}$ *holds*

$$Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} < z)) = 1$$

Proof. Let $z \in \mathbb{Z}$. By Lemma 4.14, $\mathcal{C}$ contains both a positive and negative cycle. Thus, we find (not necessarily simple) cycles $\vartheta_>$ and $\vartheta_<$ of $\mathcal{M}$ both starting in s, such that $\mathsf{weight}(\vartheta_>) > z$ and $\mathsf{weight}(\vartheta_<) < z$.

In the following let us argue for $Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} > z)) = 1$, the argument for $Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} < z)) = 1$, is analogous. Let $F_>$ be the set of all $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$, where $\mathsf{Wgt}_n(\pi) = \mathsf{weight}(\vartheta_>)$ for infinitely many $n \in \mathbb{N}$. Clearly

$$Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} > z)) \geq Pr_s^{\mathcal{M}}(F_>).$$

The existence of $\vartheta_>$ implies that we find an $n_> \in \mathbb{N}$ such that for every path $\pi \in \mathsf{Cyl}(\vartheta_>)$ we have $\pi[\ldots \uparrow n_>] = \vartheta_>$. Therefore,

$$Pr_s^{\mathcal{M}}(\mathsf{Wgt}_{n_>} = \mathsf{weight}(\vartheta_>)) \geq Pr_s^{\mathcal{M}}(\mathsf{Cyl}(\vartheta_>)) > 0.$$

Thus, by definition of recurrence

$$1 = Pr_s^{\mathcal{M}}(F_>) \leq Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} > z)).$$

As by assumption $\mathsf{weight}(\vartheta_>) > z$ this completes the proof. $\square$

Corollary 4.16. *Assume that* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ *and* $\mathcal{C}$ *contains a negative cycle. For every integer* $z \in \mathbb{Z}$

$$Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = 0,$$
$$Pr_s^{\mathcal{M}}(\Diamond(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Box\Diamond(\mathsf{weight} > z)) = 1.$$

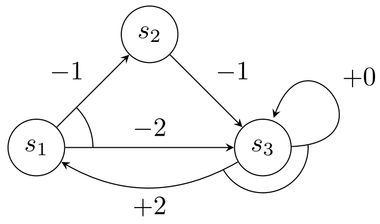

Figure 4.2: BSCC with expected weight zero and no negative cycles

Example 4.17. Consider again the Markov chain of Example 3.1 depicted in Figure 3.1. Corollary 4.16 yields that also for states s_4, s_5 and s_6 the initial battery level is irrelevant and the system almost-surely will eventually run out of battery.

Let us now assume case (b), i.e., $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does not contain negative cycles. By Lemma 4.14 this BSCC then contains only cycles with accumulated weight zero. As a consequence, for every path π there exists a finite path $\hat{\pi}$ such that $\mathsf{weight}(\pi) = \mathsf{weight}(\hat{\pi})$, as every cycle of π can be omitted. Thus, the accumulated weight for such BSCC is bounded by both the minimal and maximal weight accumulated along a finite path. Let us now formalize this intuition.

Definition 4.18. *The function $\mu : S \to \mathbb{N} \cup \pm\infty$ is defined by*

$$\mu(s) = \sup\{-\mathsf{minwgt}(\hat{\pi}) : \ \hat{\pi} \in \mathsf{FinPaths}_s^{\mathcal{M}}\}$$

Intuitively $\mu(s)$ denotes the lowest initial battery level which ensures that under no event the system will run out of battery.

Example 4.19. Even though in the following we will use the notion μ only for BSCCs with $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and do not contain negative cycles, let us consider the BSCC $\mathcal{C}$ depicted in Figure 4.1. As $\mathcal{C}$ does contain a negative cycle we have $\mu(s) = -\infty$ for every $s \in \mathcal{C}$. In fact $\mu(s) \neq -\infty$ if and only if $\mathcal{C}$ does not contain negative cycles.

Consider now the BSCC $\mathcal{C}'$ depicted in Figure 4.2. The expected weight of $\mathcal{C}'$ is zero and $\mathcal{C}'$ does not contain negative cycles. We have

$$\mu(s_1) = 2 \qquad \mu(s_2) = 1 \qquad \mu(s_3) = 0.$$

Thus, there exists no path starting in s_1 which violates $\Box(\mathsf{weight} > -3)$. However, almost all paths starting in s_1 violate $\Box(\mathsf{weight} > -2)$. Hence,

$$Pr_{s_1}^{\mathcal{M}}(\Box(\mathsf{weight} > -3)) = 1 \quad \text{and} \quad Pr_{s_1}^{\mathcal{M}}(\Box(\mathsf{weight} > -2)) = 0.$$

Lemma 4.20. *Let $\mathcal{C}$ be a BSCC of $\mathcal{M}$ such that $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and such that $\mathcal{C}$ contains no negative cycles. Let $s \in \mathcal{C}$ and $z \in \mathbb{Z}$.*

(1) If $z < -\mu(s)$, then $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = 1$.

(2) If $z \geq -\mu(s)$, then $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = 0$.

Proof. We first consider the modality $\Box$. Assume $z < -\mu(s)$. By definition of μ every finite path $\hat{\pi}$ starting from s satisfies $\mathsf{weight}(\hat{\pi}) \geq -\mu(s) > z$. This immediately yields (1) for case $\Box$, i.e., $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = 1$.

We now prove (2) for the case $\Box$. Let $z \geq -\mu(s)$. Towards a contradiction, assume that $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) > 0$. By definition of μ there exists finite path $\hat{\pi}$ starting at s such that $\mathsf{weight}(\hat{\pi}) = -\mu(s)$. As $Pr_s^{\mathcal{M}}$-almost all infinite paths of $\mathcal{M}$ contain $\hat{\pi}$, there exists an infinite path $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ such that $\pi \models \Box(\mathsf{weight} > z)$ and π contains $\hat{\pi}$. Let $\hat{\pi}_1 \in \mathsf{FinPaths}_{s\ldots s}^{\mathcal{M}}$ and $\pi_2 \in \mathsf{InfPaths}_s^{\mathcal{M}}$ such that $\pi = \hat{\pi}_1 \diamond \hat{\pi} \diamond \pi_2$. Since $\mathcal{C}$ does neither contain a positive nor negative cycle (cf. Lemma 4.14), we have $\mathsf{weight}(\hat{\pi}_1) = 0$. As $\pi \models \Box(\mathsf{weight} > z)$ we obtain the following contradiction which implies (2) for case $\Box$.

$$
\begin{aligned}
-\mu(s) \leq z &< \mathsf{minwgt}(\hat{\pi}_1 \diamond \hat{\pi}) \\
&\leq \mathsf{weight}(\hat{\pi}_1 \diamond \hat{\pi}) = \mathsf{weight}(\hat{\pi}) = -\mu(s).
\end{aligned}
$$

In order to complete the proof, we show that for every $z \in \mathbb{Z}$,

$$Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)).$$

Clearly, $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) \leq Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z))$. The reverse inequality follows from the definition of μ. Recall that $\mathsf{minwgt}(\hat{\pi})$ is defined as the minimal accumulated weight of any nonempty prefix of $\hat{\pi}$. Let $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ such that $\pi \models \Diamond\Box(\mathsf{weight} > z)$ and π visits s infinitely often (if there is no such path it trivially holds $Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) = 0 \leq Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)))$. Since $\mathcal{C}$ contains only cycles with weight zero, there exists $n \in \mathbb{N}$ such that $\pi[n\ldots] \models \Box(\mathsf{weight} > z)$ and $\mathsf{first}(\pi[n\ldots]) = s$. Reusing the arguments for (2) in the $\Box$ case, this implies $z < -\mu(s)$ as $z \geq -\mu(s)$ would contradict $\pi[n\ldots] \models \Box(\mathsf{weight} > z)$. Using (1) for $\Box$, we obtain $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = 1$ and hence

$$Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \leq 1 = Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z))$$

$\Box$

Up to now we have proven results for cases (b), (c) and (d) (cf. page 32). Let us now consider the remaining case (a), i.e., $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$.

First of all, we can directly state an analogon to Lemma 4.10.

Lemma 4.21. *Assuming $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ for $Pr_s^{\mathcal{M}}$-almost all infinite paths π of $\mathcal{M}$*

$$\sup_{n\in\mathbb{N}} \mathsf{weight}(\pi[\ldots n]) = \sup_{n\in\mathbb{N}} \mathsf{Wgt}_n(\pi) = +\infty.$$

Proof. Analogous to the proof of Lemma 4.10. $\Box$

However, in contrast to Lemma 4.10 this does not directly yield any results for $\heartsuit \in \{\Box, \Diamond\Box\}$, as we additionally need to consider the corresponding infimum. Using a transformation from weight to ratio functions, we can show that for almost all paths $\inf_{n\in\mathbb{N}} \mathsf{weight}(\pi[\ldots n]) = +\infty$, i.e., for every $z \in \mathbb{Z}$ there exists a point after which the accumulated weight does not drop below z. The proof is based on the fact that for every infinte path in a BSCC the ratio of it's prefixes tends towards the BSCC's long-run ratio.

Lemma 4.22. *Assuming $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$, for every integer $z \in \mathbb{Z}$ holds*

$$Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = 1.$$

Proof. We first show that $Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > 0)) = 1$. Let

$$\mathsf{utility} = 2 \cdot \max_{(s,s')\in S^2} \mathsf{weight}(s,s') \quad \text{and} \quad \mathsf{energy} = \mathsf{utility} - \mathsf{weight}.$$

Notice that $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ implies $\max_{(s,s')\in S^2} \mathsf{weight}(s,s') > 0$. Hence, $\mathsf{utility}$ and energy are two reward functions for $\mathcal{M}$, where energy is positive. Further, Lemma 4.1 implies $Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > 0)) = Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{ratio} > 1))$, where $\mathsf{ratio} = \mathsf{utility}/\mathsf{energy}$ is defined as usual. Thus, it suffices to prove that $Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{ratio} > 1)) = 1$. Since $\mathsf{utility} = \mathsf{energy} + \mathsf{weight}$ and $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ we obtain

$$\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = \frac{\mathbb{E}_{\mathcal{C}}(\mathsf{utility})}{\mathbb{E}_{\mathcal{C}}(\mathsf{energy})} = \frac{\mathbb{E}_{\mathcal{C}}(\mathsf{weight})}{\mathbb{E}_{\mathcal{C}}(\mathsf{energy})} + 1 > 1.$$

Since s is a state of a BSCC, for $Pr_s^{\mathcal{M}}$-almost all infinite paths $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ holds $1 < \mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = \lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n])$. Therefore, for $Pr_s^{\mathcal{M}}$-almost all paths $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$, and all $\varepsilon \in \mathbb{R}_{>0}$ there exists $n' \in \mathbb{N}$, such that for every $n \in \mathbb{N}_{\geq n'}$, $\mathsf{ratio}(\pi[\ldots n]) \geq \mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) - \varepsilon$. As there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) - \varepsilon > 1$, this implies $\pi \models \Diamond\Box(\mathsf{ratio} > 1)$ for $Pr_s^{\mathcal{M}}$-almost all infinite paths of $\mathcal{M}$.

Let $z \in \mathbb{Z}$. We now show that $Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = 1$, by arguing over $Pr_s^{\mathcal{M}}$-almost all paths $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$. As $Pr_s^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > 0)) = 1$, the following minima exist,

$$\begin{aligned} \tilde{W}(\pi) &= \min\{\mathsf{weight}(\rho) \colon \rho \text{ prefix of } \pi\}, \\ W(\pi) &= \min\{\tilde{W}(\pi[n\ldots]) \colon n \in \mathbb{N} \text{ and } \mathsf{first}(\pi[n\ldots]) = s\}. \end{aligned}$$

Applying Lemma 4.21, there exists $n \in \mathbb{N}$ such that $\mathsf{weight}(\pi[\ldots n]) > z - W(\pi)$ and $\mathsf{first}(\pi[n\ldots]) = s$. Hence, $\pi \models \Diamond\Box(\mathsf{weight} > z)$. □

Even though Lemma 4.22 does not cover modality $\Box$, we will use the lemma to prove by contradiction that there has to exist a natural number n such that $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0$.

Lemma 4.23. *Assuming $\mathbb{E}_{\mathcal{C}} > 0$, there exists $n \in \mathbb{N}$ such that*

$$Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0.$$

Proof. Towards a contradiction assume that $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) = 0$ for all $n \in \mathbb{N}$, which is equivalent to $Pr_s^{\mathcal{M}}(\Diamond(\mathsf{weight} \leq -n)) = 1$. Notice that, for every infinite path π of $\mathcal{M}$ and for every $n \in \mathbb{N}$

$$\pi \models \Diamond(\mathsf{weight} \leq -(n+1)) \qquad \text{implies} \qquad \pi \models \Diamond(\mathsf{weight} \leq -n).$$

Hence as every measure is continuous from above, we have

$$Pr_s^{\mathcal{M}}(\bigwedge_{n\in\mathbb{N}} \Diamond(\mathsf{weight} \leq -n)) = \inf_{n\in\mathbb{N}} Pr_s^{\mathcal{M}}(\Diamond(\mathsf{weight} \leq -n)) = 1.$$

Thus, for $Pr_s^{\mathcal{M}}$-almost all paths $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$, there exists a strictly monotonically increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers with $\sup_{k\in\mathbb{N}} \mathsf{weight}(\pi[\ldots n_k]) = -\infty$. This, however, implies that $Pr_s^{\mathcal{M}}$-almost all infinite paths of $\mathcal{M}$ do not satisfy $\Diamond\Box(\mathsf{weight} > 0)$ and therefore contradicts Lemma 4.22. □

However, to be able to decide the positive $\Box$-weight problem for $\mathcal{C}$, the pure existence of such an n as stated in Lemma 4.23 is not sufficient. Observe that Lemma 4.23 implies

$$Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > \mathsf{minwgt}(\vartheta)) > 0$$

for every weight-positive cycle ϑ starting in s. In fact, we can show that the existence of a cycle ϑ with $\mathsf{weight}(\vartheta) > 0$ and $\mathsf{minwgt}(\vartheta) > -n$ not only is a sufficient but also a necessary criterion for $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0$. To this end, the following lemma will be useful.

Lemma 4.24. *Assuming* $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$*, there exists a positive cycle* ϑ *of* $\mathcal{M}$ *starting from* s.

Proof. If $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$, then Lemma 4.21 yields the existence of $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$ such that $\sup_{n\in\mathbb{N}} \mathsf{Wgt}_n(\pi) = +\infty$. We thus find a $k \in \mathbb{N}$ where $\mathsf{Wgt}_k(\pi) > 0$ and thus $\pi[\ldots \uparrow k]$ is a positive cycle of $\mathcal{C}$ starting in s. □

Lemma 4.25. *Let* $n \in \mathbb{N}$*. Assuming* $\mathbb{E}_{\mathcal{C}} > 0$*, if* $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0$*, there exists a cycle* ϑ *of* $\mathcal{M}$*, starting at* s*, such that*

$$\mathsf{weight}(\vartheta) > 0 \textit{ and } \mathsf{minwgt}(\vartheta) > -n.$$

Proof. Assume that $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0$. Furthermore, we assume that for every cycle ϑ of $\mathcal{M}$ starting at s and satisfying $\mathsf{weight}(\vartheta) > 0$, the following holds,

$$\mathsf{minwgt}(\vartheta) \leq -n. \tag{4.4}$$

Lemma 4.24 states that there exists at least one positive cycle ϑ' of $\mathcal{M}$ starting at s. Thus, if we derive a contradiction from assumption (4.4), the proof of the lemma is achieved. Since $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0$ and Lemma 4.22 we find $\pi \in \mathsf{InfPaths}_s^{\mathcal{M}}$, such that π visits s infinitely often and $\pi \models \Box(\mathsf{weight} > -n) \wedge \Diamond\Box(\mathsf{weight} > 0)$. Under those assumptions for every $k \in \mathbb{N}$, $\vartheta_k = \pi[\ldots \uparrow k]$ denotes a cycle with

$\mathsf{weight}(\vartheta_k) \leq 0$ as otherwise we derive the following contradiction: $\mathsf{weight}(\vartheta_k) > 0$ implies $\mathsf{minwgt}(\vartheta_k) \leq -n$ which yields $\pi \models \Diamond(\mathsf{weight} \leq -n)$.

However, $\mathsf{weight}(\vartheta_k) \leq 0$ for all $k \in \mathbb{N}$ contradicts $\pi \models \Diamond\Box(\mathsf{weight} > 0))$, and thus assumption (4.4) yields a contradiction. This completes the proof. □

Example 4.26. Consider again the Markov chain $\mathcal{M}$ of Example 3.1 depicted in Figure 3.1. Lemma 4.25 yields that state s_1 has to be reached with battery level at least 2 to avoid that the system will eventually run out of battery almost-surely. Notice that even though a battery level of 2 is sufficient to prevent that the system breaks down almost-surely, there is no battery level to ensure that the system almost-surely does not run out of battery. No matter how high the initial battery is, there always is a non-zero probability to take the negative cycle $s_1 s_2 s_3 s_1$ often enough to completely discharge the battery.

4.3 Analyzing the underlying graph

In the previous section we investigated BSCCs and, dependent on their expected weight, the minimal n such that the positive $\heartsuit$-weight problem yields a positive answer for $\heartsuit \in \{\Box, \Diamond\Box\}$ and $z = -n$. This n can be interpreted as the minimal credit (minimal battery level, respectively) needed to ensure a positive probability for $\heartsuit(\mathsf{weight} > 0)$. Thus, the missing part to solve the positive $\heartsuit$-weight problem for $\heartsuit \in \{\Box, \Diamond\Box\}$ and $z = 0$ is the answer to the following question:

Given a state s and a minimal credit $c \in \mathbb{N}_{>0}$, is the probability of reaching state s while accumulating at least c and not violating $\heartsuit(\mathsf{weight} > 0)$ *positive?*

As we are only asking for positive probability, it suffices to show whether or not there exists a finite path from ι to s accumulating at least c and not violating $\heartsuit(\mathsf{weight} > 0)$. In this section we will introduce some basic notions for the underlying graph of $\mathcal{M}$, which we will use in the next section to prove the promised characterizations for the qualitative $\heartsuit$-weight problems.

Note, $\Diamond\Box(\mathsf{weight} > 0)$ does not impose any additional restrictions on the path reaching a BSCC. This motivates the definition of $dist_{\min}(s, s')$ and $dist_{\max}(s, s')$ as the minimal, respectively maximal, possible accumulated weight on a path from state s to state s'.

Definition 4.27. *The distance functions $dist_{\min}$, $dist_{\max}$: $S \times S \to \mathbb{N} \cup \{\pm\infty\}$ are defined as*

$$dist_{\min}(s, s') = \inf_{\hat{\pi}} \mathsf{weight}(\hat{\pi})$$
$$dist_{\max}(s, s') = \sup_{\hat{\pi}} \mathsf{weight}(\hat{\pi})$$

where $\hat{\pi}$ ranges over all paths of $\mathsf{FinPaths}^{\mathcal{M}}_{s \ldots s'}$.

Remark 4.28. Notice that $dist_{\min}(s, s') = \infty$ if and only if s' is not reachable from s, whereas $dist_{\min}(s, s') = -\infty$ if and only if there exists a path from s to s' containing a negative cycle. Similarly $dist_{\max}(s, s') = -\infty$ iff s' is not reachable from s and $dist_{\max} = +\infty$ iff there exists a path from s to s' containing a positive cycle.

Example 4.29. Consider the Markov chain $\mathcal{M}$ of Example 3.1 depicted in Figure 3.1. Let us list some values of $dist_{\min}$ and $dist_{\max}$.

$$
\begin{aligned}
dist_{\min}(s_1, s_4) &= +\infty & \qquad dist_{\max}(s_1, s_4) &= -\infty \\
dist_{\min}(s_1, s_2) &= -\infty & \qquad dist_{\max}(s_1, s_2) &= +\infty \\
dist_{\min}(\iota, t_3) &= 8 & \qquad dist_{\max}(\iota, t_3) &= +\infty
\end{aligned}
$$

In contrast to $\Diamond\Box$, the modality $\Box$ does impose a restriction on the paths to consider. Even though there might exist a path from s to s' containing a positive cycle and thus $dist_{\max}(s, s') = +\infty$, this path might violate $\Box(\mathsf{weight} > 0)$. This motivates the definition the function *cdist*, which denotes the maximal accumulated weight from ι to s while satisfying $\Box(\mathsf{weight} > z)$.

Definition 4.30.

$$cdist(s, z) = \sup_{\hat{\pi}} \mathsf{weight}(\hat{\pi}),$$

where $\hat{\pi}$ ranges over $\mathsf{FinPaths}^{\mathcal{M}}_{\iota\ldots s}[\mathsf{minwgt} > z]$.

Example 4.31. Consider again the Markov chain $\mathcal{M}$ of Example 3.1 depicted in Figure 3.1. Let us list some values for *cdist*.

$$
\begin{aligned}
cdist(t_2, 6) &= -\infty & \quad cdist(t_2, 5) &= -\infty & \quad cdist(t_2, 4) &= +\infty \\
cdist(t_3, 6) &= 8 & \quad cdist(t_3, 5) &= 8 & \quad cdist(t_3, 4) &= +\infty \\
cdist(s_1, 6) &= 8 & \quad cdist(s_1, 5) &= +\infty & \quad cdist(s_1, 4) &= +\infty
\end{aligned}
$$

4.4 A characterization for qualitative weight problems

We now have all we need to state characterizations for the qualitative $\heartsuit$-weight problems where $\heartsuit \in \{\Box, \Diamond\Box\}$. Remember that we restricted ourselves to the modalities $\Box$ and $\Diamond\Box$ as by Lemma 4.2 the remaining modalities are reducible to either of these two.

As the characterizations are based on similar ideas and the proofs for modality $\Box$ are more involved, let us start with the qualitative $\Diamond\Box$-weight problem.

4.4.1 Qualitative $\Diamond\Box$-weight problems

Remind the classification of BSCC in Section 4.2. Let $\mathcal{C}$ be a BSCC, then $\mathcal{C}$ satisfies exactly one of the following statements:

(a) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$.

(b) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ contains no negative cycle.

(c) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does contain a negative cycle.

(d) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$.

In Section 4.2 we showed that for every BSCC $\mathcal{C}$ of type (c) or (d) and any integer $z \in \mathbb{Z}$

$$Pr_{\iota}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = 0.$$

Furthermore, for every state s of a BSCC of type (a) and (b) and every integer z either almost all paths starting in s or almost no path starting in s satisfies $\Diamond\Box(\mathsf{weight} > z)$, ie

$$Pr_{s}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) \in \{0, 1\}.$$

The above observation implies the two key principles of the characterizations for the qualitative $\Diamond\Box$-weight problems:

(I) The positive $\Diamond\Box$-weight problem yields a positive answer if and only if for at least one state s of a type (a) or type (b) BSCC there exists a path $\hat{\pi}$ from ι to s accumulating *enough* weight.

(II) The almost-sure $\Diamond\Box$-weight problem yields a positive answer if and only if all reachable BSCC are of type (a) or type (b) and all paths reaching a BSCC accumulate *enough* weight.

The results of Section 4.2 also imply what *enough* means. Lemma 4.22 yields that for a BSCC of type (a) the accumulated weight from ι to s is irrelevant, ie. every weight is *enough*, whereas if s is contained in a BSCC of type (b) by Lemma 4.20 *enough* stands for $\mu(s) + z$. In the following we will state formal proofs for both qualitative $\Diamond\Box$-weight problems, which are based exactly on these principles.

Lemma 4.32. $Pr_{\iota}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) > 0$ *iff there exists a BSCC* $\mathcal{C}$ *of* $\mathcal{M}$ *for which one of the following cases holds:*

(1) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$.

(2) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$*,* $\mathcal{C}$ *contains no negative cycles, and for some* $s \in \mathcal{C}$ *holds* $\mu(s) + z < dist_{\max}(\iota, s)$*.*

Proof. We first show that the existence of a reachable BSCC $\mathcal{C}$ satisfying either (1) or (2) implies $Pr_{\iota}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) > 0$. For case (1), Lemma 4.22 directly implies the claim. Assume that $\mathcal{C}$ satisfies (2). As there exists $s \in \mathcal{C}$ and $\hat{\pi} \in \mathsf{FinPaths}_{\iota}^{\mathcal{M}}$ such that $\mathsf{last}(\hat{\pi}) = s$ and $\mu(s) + z < \mathsf{weight}(\hat{\pi})$, using Lemma 4.20 we have $Pr_{s}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z - \mathsf{weight}(\hat{\pi}))) = 1$ and thus

$$Pr_{\iota}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) \geq P(\hat{\pi}) \cdot Pr_{s}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z - \mathsf{weight}(\hat{\pi}))) > 0.$$

Let us now show the reverse implication. To this end, assume that every reachable BSCC of $\mathcal{M}$ neither satisfies (1) nor (2). Thus, for every BSCC $\mathcal{C}$ of $\mathcal{M}$ one of the following holds (remember Lemma 4.14):

(a) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$

(b) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ contains a negative cycle

(c) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$, $\mathcal{C}$ does not contain a negative cycle, and for every $s \in \mathcal{C}$ holds $\mu(s) + z \geq dist_{\max}(\iota, s)$.

Given a BSCC $\mathcal{C}$ which satisfies (a) or (b), Lemma 4.10 and Lemma 4.13 directly yield $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = 0$. Assume that $\mathcal{C}$ satisfies (c). Lemma 4.20 implies that for every $\hat{\pi} \in \mathsf{InfPaths}_\iota^{\mathcal{M}}$ with $\mathsf{last}(\hat{\pi}) \in \mathcal{C}$ and $\mu(\mathsf{last}(\hat{\pi})) + z \geq \mathsf{weight}(\hat{\pi})$ holds $Pr_{\mathsf{last}(\hat{\pi})}^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z - \mathsf{weight}(\hat{\pi}))) = 0$. Since every finite path $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ reaching $\mathcal{C}$ satisfies $\mu(\mathsf{last}(\hat{\pi})) + z \geq \mathsf{weight}(\hat{\pi})$, applying Lemma 4.20 we have $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C})) = 0$. Thus,

$$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = \sum_{\mathcal{C}} Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = 0$$

where $\mathcal{C}$ ranges over all reachable BSCCs of $\mathcal{M}$. □

Lemma 4.33. *$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = 1$ if and only if for each BSCC $\mathcal{C}$ of $\mathcal{M}$ one of the following cases holds:*

(1) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$.

(2) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$*, $\mathcal{C}$ contains no negative cycles, and for every $s \in \mathcal{C}$ holds $\mu(s) + z < dist_{\min}(\iota, s)$.*

Proof. The argument is analogous to the proof of Lemma 4.32. We have

$$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) = \sum_{\mathcal{C}} Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C})$$

where $\mathcal{C}$ ranges over all reachable BSCCs of $\mathcal{M}$. Using Lemma 4.22 and Lemma 4.20, we can show that for all BSCCs satisfying either (1) or (2) holds

$$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = Pr_\iota^{\mathcal{M}}(\Diamond\mathcal{C}).$$

Let $\mathcal{C}$ be a reachable BSCC neither satisfying (1) nor (2). Using Corollary 4.11, Corollary 4.16, Lemma 4.20 and similar arguments as presented in the proof of Lemma 4.32 we have

$$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = 0.$$

The fact $\sum_{\mathcal{C}} Pr_\iota^{\mathcal{M}}(\Diamond\mathcal{C}) = 1$, where $\mathcal{C}$ ranges over all reachable BSCCs of $\mathcal{M}$, completes the proof. □

4.4.2 Qualitative $\Box$-weight problems

As mentioned before the argument for the $\Box$-weight problem is more involved. However, this only applies to the positive $\Box$-weight problem. The almost-sure $\Box$-quantile can be easily computed using the following lemma stated in [4].

Lemma 4.34. *$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z)) = 1$ iff $dist_{\min}(\iota, s) > z$ for all states s.*

Let us now consider the positive $\Box$-weight problem. The basic idea is similar as for $\Diamond\Box$. As before we look for paths reaching a BSCC of type (a) or type (b) while accumulating *enough* weight. However, we additionally have to consider the restriction that these paths have to satisfy $\Box(\mathsf{weight} > z)$. Notice that this exactly motivated the introduction of *cdist* in the previous section. In fact for BSCCs of type (b) it suffices to consider *cdist* instead of $dist_{\max}$.

Let us now consider BSCCs of type (a), i.e., with positive expected weight. To this end, let $\mathcal{C}$ be such a BSCC and $s \in \mathcal{C}$. In contrast to $\Diamond\Box$ there exists a maximal $z \in \mathbb{Z}$ such that

$$Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z)) > 0. \tag{4.5}$$

By Lemma 4.25 the maximal z satisfying (4.5) is the maximum value of $\mathsf{minwgt}(\vartheta)$ where ϑ ranges over all positive cycles starting in s. In the following we will denote this value by $z_{\max}$ and the associated positive cycle by $\vartheta_{\max}$. Thus, *enough* with respect to the accumulated path from ι to s would be $z - z_{\max} + 1$. Notice that $cdist(s, z) \geq z - z_{\max} + 1$ implies $cdist(s, z) = +\infty$. This trivially is the case if $z_{\max} \geq 0$ as $cdist(s, z) > z$ and thus there exists a path from ι to s containing a positive cycle (namely $\vartheta_{\max}$) and satisfying $\Box(\mathsf{weight} > z)$, i.e., $cdist(s, z) = +\infty$. Assume $z_{\max} < 0$. Thus, there exists a path $\hat{\pi}$ from ι to s satisfying $\Box(\mathsf{weight} > z)$ and accumulating $z + |z_{\max}| + 1$ and $\hat{\pi} \diamond \vartheta$ is a witness for $cdist(s, z) = +\infty$.

Putting this together, this yields the following characterization for the positive $\Box$-weight problem.

Lemma 4.35. *$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z)) > 0$ iff there exists a BSCC $\mathcal{C}$ with $s \in \mathcal{C}$ such that one of the following two conditions holds:*

(1) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ and $cdist(s, z) = +\infty$, or

(2) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$, $\mathcal{C}$ contains no negative cycles, and $\mu(s) + z < cdist(s, z)$.

We dedicate the remainder of this section to the formal proof of this lemma. It follows directly from Lemma 4.10 and Lemma 4.13 that the BSCC $\mathcal{C}$ necessarily has to be of either type (a) or type (b), as for any other BSCC $\mathcal{C}$ and arbitrary $z \in \mathbb{Z}$ we have $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = 0$. Hence the following two lemmata directly yield the correctness of the above characterization.

Lemma 4.36. *Let $\mathcal{C}$ be a BSCC of the Markov chain $\mathcal{M}$ such that $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$. $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) > 0$ iff there exists $s \in \mathcal{C}$ such that $cdist(s, z) = +\infty$.*

Lemma 4.37. *Let $\mathcal{C}$ be a BSCC of $\mathcal{M}$ such that $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ contains no negative cycles. $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) > 0$ if and only if there exists $s \in \mathcal{C}$ such that $\mu(s) + z < cdist(s, z)$.*

Proof of Lemma 4.36. We show two implications. We first assume that $s \in \mathcal{C}$ satisfies $cdist(s, z) = +\infty$, and argue that $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C})$ is positive.

By Lemma 4.23 we know that there exists $n \in \mathbb{N}$ with $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0$. Since $cdist(s, z) = +\infty$, there exists a finite path $\hat{\pi} \in \mathsf{FinPaths}_{\iota\ldots s}^{\mathcal{M}}[\mathsf{minwgt} > z]$, such that $\mathsf{weight}(\hat{\pi}) \geq z + n$. Thus,

$$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) \geq P(\hat{\pi}) \cdot Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -n)) > 0,$$

which yields the first implication.

To show the remaining implication, assume $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) > 0$. Thus,

$$0 < Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z) \wedge \Diamond\mathcal{C}) = \sum_{\hat{\pi}} P(\hat{\pi}) \cdot Pr_{\mathsf{last}(\hat{\pi})}^{\mathcal{M}}(\Box(\mathsf{weight} > -\mathsf{weight}(\hat{\pi}) + z)),$$

where $\hat{\pi}$ ranges over all finite paths of $\mathcal{M}$ starting from ι such that $\mathsf{last}(\hat{\pi})$ is the only state of $\hat{\pi}$ contained in $\mathcal{C}$ and $\mathsf{minwgt}(\hat{\pi}) > z$. Hence, there exists $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}$, where $\mathsf{last}(\hat{\pi}) = s \in \mathcal{C}$, $\mathsf{minwgt}(\hat{\pi}) > z$, and $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -\mathsf{weight}(\hat{\pi}) + z)) > 0$. Lemma 4.25 yields the existence of a cycle ϑ of $\mathcal{M}$, starting from s, such that $\mathsf{weight}(\vartheta) > -\mathsf{weight}(\hat{\pi}) + z$. For every $n \in \mathbb{N}_{>0}$ define $\hat{\pi}_n \in \mathsf{FinPaths}^{\mathcal{M}}$ by $\hat{\pi}_n = \hat{\pi} \diamond (\vartheta)^n$, where $\vartheta^m = \vartheta \diamond \vartheta^{m-1}$. Clearly, for every $n \in \mathbb{N}$, $\hat{\pi}_n$ is a finite path in $\mathsf{FinPaths}_{\iota\ldots s}^{\mathcal{M}}[\mathsf{minwgt} > z]$. Additionally for every natural number n holds $\mathsf{weight}(\hat{\pi}_n) < \mathsf{weight}(\hat{\pi}_{n+1})$ as $\mathsf{weight}(\vartheta) > 0$. This yields $cdist(s, z) = +\infty$. $\square$

Proof of Lemma 4.37. We show both directions separately.

($\Leftarrow$). Assume that s is a state of $\mathcal{C}$ which satisfies $\mu(s) + z < cdist(s, z)$. Thus, there exists $\hat{\pi} \in \mathsf{FinPaths}_{\iota\ldots s}^{\mathcal{M}}[\mathsf{minwgt} > z]$, such that $\mu(s) + z < \mathsf{weight}(\hat{\pi})$. Thus, $-\mathsf{weight} + z < -\mu(s)$. Lemma 4.20 yields $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -\mathsf{weight}(\hat{\pi}) + z)) = 1$ which completes the argument, as

$$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > 0) \wedge \Diamond\mathcal{C}) \geq P(\hat{\pi}) \cdot Pr_{\mathsf{last}(\hat{\pi})}^{\mathcal{M}}(\Box(\mathsf{weight} > -\mathsf{weight}(\hat{\pi}) + z)) > 0.$$

($\Rightarrow$). We show the contraposition. To this end, assume that for every $s \in \mathcal{C}$ holds $cdist(s, z) \leq \mu(s) + z$. Then we obtain for every state $s \in \mathcal{C}$ and finite path $\hat{\pi} \in \mathsf{FinPaths}_{\iota\ldots s}^{\mathcal{M}}[\mathsf{minwgt} > z]$

$$\mathsf{weight}(\hat{\pi}) \leq cdist(s, z) \leq \mu(s) + z.$$

However, by Lemma 4.20 this implies $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -\mathsf{weight}(\hat{\pi}) + z)) = 0$ which yields the claim as

$$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > 0) \wedge \Diamond\mathcal{C}) \leq \sum_{\hat{\pi}} P(\hat{\pi}) \cdot Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > -\mathsf{weight}(\hat{\pi}) + z)) = 0,$$

where the sum ranges over all finite paths $\hat{\pi} \in \bigcup_{s \in \mathcal{C}} \mathsf{FinPaths}_{\iota\ldots s}^{\mathcal{M}}[\mathsf{minwgt} > z]$. $\square$

4.5 Deciding qualitative weight problems in polynomial time

Keeping the previous sections' characterizations in mind, it seems natural to investigate the computational complexity of $dist_{\min}$, $dist_{\max}$, $cdist(\cdot, z)$ and μ. In this section we will show that all these functions are computable in polynomial time.

Before we do so, let us first discuss why this yields a polynomial decision procedure for the qualitative weight problems. Lemma 4.2 implies that any qualitative $\Diamond$-weight problem can be transformed into a qualitative $\Box$-weight problem in polynomial time and analogously any $\Box\Diamond$-weight problem can be reduced polynomially to a qualitative $\Diamond\Box$-weight problem. As in the previous sections, we can hence restrict ourselves to qualitative $\heartsuit$-weight problems where $\heartsuit \in \{\Box, \Diamond\Box\}$.

Without loss of generality, in the following we assume that $\mathcal{M}$ is a Markov chain containing only states which are reachable from ι. Lemma 4.34 directly yields that if $dist_{\min}$ is computable in polynomial time, the almost-sure $\Box$-weight problem is decidable in polynomial time. Let us now consider the positive $\Box$-weight problem. The argument for qualitative $\Diamond\Box$-weight problems is analogous. Assuming that $dist_{\min}$, $dist_{\max}$, $cdist(\cdot, z)$ and μ are computable in polynomial time, Lemma 4.35 yields the following polynomial-time decision procedure for the almost-sure $\Box$-weight problem:

1. Compute the BSCCs of $\mathcal{M}$ and their expected weight, which can be done in polynomial time using the results stated in Chapter 2
2. Compute μ for every state of a BSCC $\mathcal{C}$ with $\mathbb{E}_{\mathcal{C}} = 0$.
3. Classify the BSCCs into the four types of Section 4.2 (cf. page 32). As mentioned in Example 4.19, the values of μ can be used to decide whether a BSCC contains a negative cycle.
4. Compute $dist_{\min}$ and $dist_{\max}$ for every state contained in a BSCC $\mathcal{C}$ of type (a) or type (b), i.e., with either $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ or $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does not contain negative cycles.
5. Compute $cdist$ for very state contained in a BSCC of type (a) or type (b).
6. Check if the conditions of Lemma 4.35 are satisfied. When considering qualitative $\Diamond\Box$-weight problems consider Lemma 4.32 or Lemma 4.33, respectively.

Notice that, motivated by the aim to give a similar decision procedure for $\Box$ and $\Diamond\Box$, the stated procedure contains some redundant computations. For modality $\Box$ the third step could be omitted, whereas for modality $\Diamond\Box$ one does not need to compute $cdist$.

In the remainder of this section we will show that $dist_{\min}$, $dist_{\max}$, $cdist(\cdot, z)$ and μ are in fact computable in polynomial time.

Notice that, if we consider the Markov chain $\mathcal{M}$ as a weighted graph, $dist_{\min}(s, s')$ for $s, s' \in S$ is equivalent to the length of the shortest path from s to s'. Hence,

$\mathit{dist}_{\min}$ is computable in polynomial time in the size of $\mathcal{M}$ using ordinary shortest path algorithms for weighted graphs, e.g., Bellman-Ford. Further notice that $\mathit{dist}_{\max}$ for weight is equivalent to $\mathit{dist}_{\min}$ for $-\mathsf{weight}$. This yields the following lemma.

Lemma 4.38. *The functions $\mathit{dist}_{\min}$ and $\mathit{dist}_{\max}$ are computable in time polynomial in the size of $\mathcal{M}$.*

To compute $\mu(s)$ we can use the values of $\mathit{dist}_{\min}$. Remind that $\mu(s)$ is defined as $\mu(s) = \inf\{k \in \mathbb{N} \colon s \models \forall\Box(\mathsf{weight} > -k)\}$. Thus $\mu(s)$ must be greater than $-\mathsf{weight}(\hat{\pi})$ for every $\hat{\pi}$ starting in s, i.e.,

$$\mu(s) - 1 \geq -\min_{s' \in \mathcal{C}} \mathit{dist}_{\min}(s, s').$$

Formally, the definition of μ yields $\mu(s) \geq -\min_{s' \in \mathcal{C}} \mathit{dist}_{\min}(s, s') + 1$. Furthermore, $\mu(s) > k$ implies that there exists a state $\tilde{s}$ and a finite path $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}_{s \ldots \tilde{s}}$ such that $\mathsf{weight}(\hat{\pi}) \leq -k$. Therefore, by definition of $\mathit{dist}_{\min}$ we have

$$\mathsf{weight}(\hat{\pi}) \geq \mathit{dist}_{\min}(s, \tilde{s}) \geq \min_{s' \in \mathcal{C}} \mathit{dist}_{\min}(s, s').$$

Thus, the assumption $\mu(s) > -\min_{s \in \mathcal{C}} \mathit{dist}_{\min}(s, s') + 1$ yields a contradiction as $-\min_{s' \in \mathcal{C}} \mathit{dist}_{\min}(s, s') > -\infty$ contradicts

$$\min_{s' \in \mathcal{C}} \mathit{dist}_{\min}(s, s') \leq \min_{s' \in \mathcal{C}} \mathit{dist}_{\min}(s, s') - 1.$$

Hence, we can conclude the following lemma.

Lemma 4.39. *For every state s of a BSCC the function μ is computable in time polynomial in the size of $\mathcal{M}$.*

Now the only missing part for a polynomial-time decision procedure for qualitative weight problems is the computation of cdist. The remainder of this section is dedicated to prove that for any $s \in S$ and $z \in \mathbb{Z}$ the value $\mathit{cdist}(s, z)$ can be computed in polynomial time. To this end, we will present a modified Bellman-Ford algorithm, which computes $\mathit{cdist}(s, z)$ for every state s of $\mathcal{M}$.

Lemma 4.40. *The function cdist is computable in polynomial time in the size of $\mathcal{M}$.*

Proof. As mentioned before, we prove the claim by presenting a modified Bellman-Ford algorithm that, given a weighted Markov chain $\mathcal{M} = (S, \iota, P, \mathsf{weight})$ and $z \in \mathbb{Z}$ as input, returns a function $\Delta \colon S \to \mathbb{Z} \cup \{\pm\infty\}$ and showing that $\Delta(s) = \mathit{cdist}(s, z)$ for every $s \in S$.

This variant works as follows. Initially, we set $\Delta_0(\iota) = 0$ and $\Delta_0(s) = -\infty$ for all remaining states s. For $i = 1, 2, \ldots, n+1$ where $n = |S|$ we consider all states $s \in S$ and compute

$$\Delta_i(s) = \max\{\Delta_{i-1}(s), \max_{s'}\{\Delta_{i-1}(s') + \mathsf{weight}(s', s)\}\}$$

where s' ranges over all predecessors of s with $\Delta_{i-1}(s') + \mathsf{weight}(s, s') > z$. In a post-processing step we use Δ_n and Δ_{n+1} to compute the values of $\Delta(s)$. We set $\Delta(s) = +\infty$ if s is reachable from some state $t \in S$ such that $\Delta_n(t) \neq \Delta_{n+1}(t)$ and $\Delta(s) = \Delta_n(s)$ otherwise.

The pseudo code of the algorithm is shown on page 48 and uses the following notations: For every $s \in S$ let $\mathsf{succ}(s)$ be the set of all direct successors of s and $\mathsf{reach}(s)$ the set of all states reachable from s, i.e.,

$$\begin{aligned} \mathsf{succ}(s) &= \{s' \in S \colon P(s, s') > 0\}, \\ \mathsf{reach}(s) &= \{s' \in S \colon s' \text{ reachable from } s \text{ in } \mathcal{M}\}. \end{aligned}$$

We now prove $\mathit{cdist}(s, z) = \Delta(s)$ for all $s \in S$ and $z \in \mathbb{Z}$. To this end, let us fix a weighted Markov chain $\mathcal{M} = (S, \iota, P, \mathsf{weight})$, and an integer $z \in \mathbb{Z}$. In what follows, n is defined to be the cardinality of S. Furthermore, for every $k \in \mathbb{N}$ and $s \in S$ the set $F(s, k)$ is given as follows: $F(s, k)$ contains all $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}[\mathsf{minwgt} > z]$ such that $|\hat{\pi}| \leq k$.

It suffices to show the following two claims:

(a) For all $s \in S$, and $k \in \mathbb{N}$ holds $\Delta_k(s) = \sup_{\hat{\pi} \in F(s,k)} \mathsf{weight}(\hat{\pi})$.

(b) For all $s \in S$, $\mathit{cdist}(s, z) = +\infty$ if and only if there exists $s' \in S$ such that $s \in \mathsf{reach}(s')$ and $\Delta_n(s') \neq \Delta_{n+1}(s')$.

Let us see why by fixing a state $s \in S$ and considering two cases, $\mathit{cdist}(s, z) < \infty$ and $\mathit{cdist}(s, z) = \infty$. In the latter $\Delta(s) = \infty$ follows directly from (b) and the post-computation step of Algorithm 1. Assume $\mathit{cdist}(s, z) < \infty$. This implies that there does not exist a path in $\mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}[\mathsf{minwgt} > z]$ that contains a positive cycle and therefore for every path $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}[\mathsf{minwgt} > z]$ there exists a path $\hat{\pi}' \in F(s, n)$ such that $\mathsf{weight}(\hat{\pi}) \leq \mathsf{weight}(\hat{\pi}')$, as one can omit the non-positive cycles of $\hat{\pi}$ to find $\hat{\pi}'$. Furthermore, we have $F(s, n) \subseteq \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}[\mathsf{minwgt} > z]$. Thus if $\mathit{cdist}(s, z) < \infty$ we have $\mathit{cdist}(s, z) = \sup_{\hat{\pi} \in F(s,n)} \mathsf{weight}(\hat{\pi})$, which using (a) implies $\mathit{cdist}(s, z) = \Delta_n(s)$. Inspecting Algorithm 1 and due to (b) it holds $\Delta_n(s) = \Delta(s)$ and therefore $\mathit{cdist}(s, z) = \Delta(s)$.

Proof of (a). The proof can be obtained by a standard induction on k. For $k = 0$ the claim is trivial. Let $k \in \mathbb{N}$ and $s \in S$. As induction hypothesis assume that for k

$$\Delta_k(s) = \sup_{\hat{\pi} \in F(s,k)} \mathsf{weight}(\hat{\pi}).$$

Using the induction hypothesis and the update condition of Algorithm 1 we obtain

$$\Delta_{k+1}(s) \leq \sup_{\hat{\pi} \in F(s,k+1)} \mathsf{weight}(\hat{\pi}).$$

In order to prove the reverse inequality, let $\hat{\pi} \in F(s, k+1)$. We can safely assume that $F(s, k+1) \neq \varnothing$, as otherwise $\Delta_{k+1}(s) = -\infty = \sup_{\hat{\pi} \in F(s,k+1)} \mathsf{weight}(\hat{\pi})$. It suffices to show that $\mathsf{weight}(\hat{\pi}) \leq \Delta_{k+1}(s)$. If $|\hat{\pi}| \leq k$ this is a direct consequence

input : weighted Markov chain $\mathcal{M} = (S, \iota, P, \mathsf{weight})$, $z \in \mathbb{Z}$
output: $\Delta \colon S \to \mathbb{N} \cup \{-\infty, \infty\}$

```
set n = |S| + 1;
set Δ_0(s) = −∞ for all s ∈ S \ {ι};
set Δ_0(ι) = 0;

for 1 ≤ i ≤ n + 1 do
    set Δ_i(s) = Δ_{i−1}(s) for all s ∈ S;
    for s ∈ S whereΔ_{i−1}(s) > −∞ do
        for s′ ∈ succ(s) do
            if Δ_{i−1}(s) + weight(s, s′) > z then
                setΔ_i(s′) = max{Δ_i(s′), Δ_{i−1}(s) + weight(s, s′)};
            end
        end
    end
end

set Δ(s) = Δ_n(s) for all s ∈ S;

for s′ ∈ S do
    if Δ_n(s′) ≠ Δ_{n+1}(s′) then
        set Δ = +∞ for all s ∈ reach(s′);
    end
end

return Δ
```

Algorithm 1: Modified Bellman-Ford algorithm of the induction hypothesis, if $|\hat{\pi}| = k+1$ we can argue as follows. Let $\hat{\pi}'$ be a finite path of $\mathcal{M}$ and $s' \in S$ such that $\hat{\pi} = \hat{\pi}' \diamond s's$. Since $\Delta_k(s') \geq \mathsf{weight}(\hat{\pi}')$ this completes the proof of (a) as

$$\Delta_{k+1}(s) \geq \Delta_k(s') + \mathsf{weight}(s', s) \geq \mathsf{weight}(\hat{\pi}') + \mathsf{weight}(s', s) = \mathsf{weight}(\hat{\pi}).$$

Proof of (b). We show the two implications separately.

($\Leftarrow$). Let $s \in S$ and assume there exists a $s' \in S$ such that $s \in \mathsf{reach}(s')$ and $\Delta_n(s') \neq \Delta_{n+1}(s')$. Clearly, $\Delta_{n+1}(s') \neq -\infty$ as statement (a) for this case implies $\Delta_{n+1}(s') > \Delta_n(s')$. Hence, there exists $\hat{\pi} \in F(s', n+1) \setminus F(s', n)$ such that $\mathsf{weight}(\hat{\pi}) = \Delta_{n+1}(s')$. As $|\hat{\pi}| = n+1$ and $\mathsf{weight}(\hat{\pi}) > \sup_{\hat{\pi}' \in F(s', n)}$ the path $\hat{\pi}$ must contain a positive cycle ϑ. Since s is reachable from s' we conclude $\mathit{cdist}(s, z) = \infty$.

($\Rightarrow$). Assume that $\mathit{cdist}(s, z) = \infty$. This implies that there exists a finite path $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}[\mathsf{minwgt} > z]$ which contains a positive cycle $\vartheta = t_0 t_1 \ldots t_m t_0$ and we can safely assume that $\hat{\pi}' \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots t_0}[\mathsf{minwgt} > z]$ is a simple path such that $\hat{\pi}' \diamond \vartheta$ is a prefix of $\hat{\pi}$. As $\hat{\pi}' \in F(t_0, n)$ claim (a) yields $\Delta_n(t_0) \neq -\infty$. Towards a contradiction, assume that for all $s' \in S$ the following implication is true:

$$s \in \mathsf{reach}(s') \text{ implies } \Delta_n(s') = \Delta_{n+1}(s').$$

Thus, in particular $\Delta_n(t_i) = \Delta_{n+1}(t_i)$ for all states t_i on ϑ. Let $s' \in S$ for which $\Delta_n(s') = \Delta_{n+1}(s')$. Inspecting Algorithm 1, we obtain for every state $s'' \in S$ with $s' \in \mathsf{succ}(s'')$

$$\Delta_n(s') = \Delta_{n+1}(s') \geq \Delta_n(s'') + \mathsf{weight}(s'', s')$$

which in particular implies

$$\Delta_n(t_0) \geq \Delta_n(t_m) + \mathsf{weight}(t_m, t_0)$$
$$\Delta_n(t_i) \geq \Delta_n(t_{i-1}) + \mathsf{weight}(t_{i-1}, t_i)$$

for all $1 \leq i \leq m$. Putting this together, we obtain the following contradiction:

$$\Delta_n(t_0) \neq -\infty \quad \text{and} \quad \Delta_n(t_0) \geq \Delta_n(t_0) + \mathsf{weight}(\vartheta).$$

□

As already discussed at the beginning of this section the lemmata of this section in combination with the characterizations of the previous section (Lemma 4.32, Lemma 4.33, Lemma 4.34 and Lemma 4.35) and the dualities of Lemma 4.2 yield the following theorem.

Theorem 4.41. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ be a weighted Markov chain, $\phi = (2^{AP})^\omega$ and $z \in \mathbb{Z}$. For $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ the qualitative $\heartsuit$-weight problems with respect to $\mathcal{M}$, ϕ and z are decidable in time polynomial in the size of $\mathcal{M}$.*

Corollary 4.42. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ be an energy-utility Markov chain, $\phi = (2^{AP})^\omega$ and $z \in \mathbb{Z}$. For $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ the qualitative $\heartsuit$-ratio problems with respect to $\mathcal{M}$, ϕ and z are decidable in time polynomial in the size of $\mathcal{M}$.*

Proof. Directly follows from Lemma 4.1. □

5 From decision problems to quantiles

In the previous chapter we have shown that, for a given threshold $z \in \mathbb{Z}$ and the side constraint $\phi = (2^{AP})^\omega$ the qualitative weight and ratio decision problems are solvable in polynomial time. In this chapter we will investigate the corresponding computational problem to determine both qualitative weight and ratio quantiles for the side constraint $\phi = (2^{AP})^\omega$. In Section 5.1 we will use the results of the previous chapter to show that qualitative weight quantiles are computable in polynomial time. Section 5.2 is dedicated to qualitative ratio quantiles, which in contrast to the decision problems for ratios cannot be deduced from the results for weighted Markov chains, but require a more involved argument.

As for the previous chapter, we generalize our results towards arbitrary omega-regular properties in Chapter 6. Thus, for the sake of short notation, we will omit the atomic propositions and labelling function when denoting Markov chains.

5.1 Weight quantiles

Let us first consider weight quantiles and for the remainder of this section fix a weighted Markov chain $\mathcal{M} = (S, P, \iota, \mathsf{weight})$. The following lemma shows that the $\Diamond$- and $\Box\Diamond$-weight quantiles can be reduced to the $\Box$- and $\Diamond\Box$-weight quantiles, respectively. Therefore, it suffices to consider qualitative $\heartsuit$-weight quantiles, where $\heartsuit \in \{\Box, \Diamond\Box\}$.

Lemma 5.1. *If the weight function* weight' *for* $\mathcal{M}$ *is given by* $\mathsf{weight}' = -\mathsf{weight}$*, then*

(1) $\mathsf{Qu}^{>0}[\Diamond\mathsf{weight}] = -(\mathsf{Qu}^{=1}[\Box\mathsf{weight}'] + 2)$,

(2) $\mathsf{Qu}^{=1}[\Diamond\mathsf{weight}] = -(\mathsf{Qu}^{>0}[\Box\mathsf{weight}'] + 2)$,

(3) $\mathsf{Qu}^{>0}[\Box\Diamond\mathsf{weight}] = -(\mathsf{Qu}^{=1}[\Diamond\Box\mathsf{weight}'] + 2)$,

(4) $\mathsf{Qu}^{=1}[\Box\Diamond\mathsf{weight}] = -(\mathsf{Qu}^{>0}[\Diamond\Box\mathsf{weight}'] + 2)$.

Proof. We prove only (1) as the remaining equalities can be proven in exactly the same manner. Using the the fact that $\sup A = -\inf\{-a : a \in A\}$ for $A \subseteq \mathbb{Z}$ and the duality $Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{weight} > z)) = 1 - Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > -(z+1)))$ (cf. Lemma 4.2), we obtain

$$\begin{aligned}\mathsf{Qu}^{>0}[\Diamond\mathsf{weight}] &= \sup\{z \in \mathbb{Z}\colon Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{weight} > z)) > 0\}\\ &= \sup\{z \in \mathbb{Z}\colon 1 - Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > -(z+1))) > 0\}\\ &= \sup\{z \in \mathbb{Z}\colon Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > -z-1)) < 1\}\\ &= -\inf\{z \in \mathbb{Z} : Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > z-1)) < 1\}.\end{aligned}$$

Let A and B be subsets of $\mathbb{Z}$ given by

$$\begin{aligned}A &= \{z \in \mathbb{Z}\colon Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > z-1)) < 1\},\\ B &= \{z \in \mathbb{Z}\colon Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > z-1)) = 1\}.\end{aligned}$$

Clearly A and B are disjoint, $A \cup B = \mathbb{Z}$ and for every $b \in B$ and $a \in A$ holds $a > b$. Hence $\inf A = \sup B + 1$ which yields

$$\begin{aligned}\mathsf{Qu}^{>0}[\Diamond\mathsf{weight}] &= -\inf\{z \in \mathbb{Z} : Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > z-1)) < 1\}\\ &= -\inf A = -(\sup B + 1)\\ &= -(\sup\{z \in \mathbb{Z}\colon Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > z-1)) = 1\} + 1)\\ &= -(\sup\{z \in \mathbb{Z}\colon Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}' > z)) = 1\} + 1 + 1)\\ &= -(\mathsf{Qu}^{=1}[\Box\mathsf{weight}'] + 2)\end{aligned}$$

$\Box$

Notice that Lemma 4.34 enables us to compute the almost-sure $\Box$-weight quantile as it implies the following lemma.

Lemma 5.2.

$$\mathsf{Qu}^{=1}[\Box\mathsf{weight}] = \min_{s \in S} dist_{\min}(\iota, s)$$

Keeping the results of Chapter 4 in mind, it seems natural to reconsider the BSCC classification introduced in Section 4.2. Given a BSCC $\mathcal{C}$, exactly one of the following statements holds

(a) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$,

(b) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does not contain a negative cycle,

(c) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does contain a negative cycle, or

(d) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) < 0$.

In the following we denote by $\mathfrak{C}_a$ the set of a BSSCs of $\mathcal{M}$ which satisfy (a). The sets $\mathfrak{C}_b$, $\mathfrak{C}_c$ and $\mathfrak{C}_d$ are defined analogously. The set of all BSCCs is denoted by $\mathfrak{C}$.

In fact, not only the almost-sure $\Box$-weight quantile but also both qualitative $\Diamond\Box$-weight quantiles can be computed using the results of the previous chapter. The characterizations for $\Diamond\Box$ stated in Lemma 4.32 and Lemma 4.33 yield the following two lemmata.

Lemma 5.3. *Let* $q_{\max} = \max_{s\in\mathfrak{C}_b}(dist_{\max}(\iota, s) - \mu(s) - 1)$.

$$\mathsf{Qu}^{>0}[\Diamond\Box\mathsf{weight}] = \begin{cases} +\infty & \text{if } \mathfrak{C}_a \neq \varnothing \\ q_{\max} & \text{if } \mathfrak{C}_a = \varnothing \text{ and } \mathfrak{C}_b \neq \varnothing \\ -\infty & \text{otherwise} \end{cases}$$

Proof. The claim is a direct consequence of Lemma 4.32. If $\mathfrak{C}_a \neq \varnothing$, there exists a BSCC $\mathcal{C}$ such that $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ and by Lemma 4.32 for every $z \in \mathbb{Z}$ we have $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) > 0$ and thus $\mathsf{Qu}^{>0}[\Diamond\Box\mathsf{weight}] = \infty$.

Assume $\mathfrak{C}_a = \varnothing$ and $\mathfrak{C}_b \neq \varnothing$. Then by Lemma 4.32 we $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) > 0$ for all $z \in \mathbb{Z}$ for which there exists an $s \in \mathfrak{C}_b$ such that $z < dist_{\max}(\iota, s) - \mu(s)$. Hence,

$$\mathsf{Qu}^{>0}[\Diamond\Box\mathsf{weight}] = \max_{s\in\mathfrak{C}_b}(dist_{\max}(\iota, s) - \mu(s) - 1).$$

If both $\mathfrak{C}_a$ and $\mathfrak{C}_b$ are empty, Lemma 4.32 implies that there exists no $z \in \mathbb{Z}$ such that $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{weight} > z)) > 0$ and therefore $\mathsf{Qu}^{>0}[\Diamond\Box\mathsf{weight}] = -\infty$. □

Lemma 5.4. *Let* $q_{\min} = \min_{s\in\mathfrak{C}_b}(dist_{\min}(\iota, s) - \mu(s) - 1)$.

$$\mathsf{Qu}^{=1}[\Diamond\Box\mathsf{weight}] = \begin{cases} -\infty & \text{if } \mathfrak{C}_c \cup \mathfrak{C}_d \neq \varnothing \\ +\infty & \text{if } \mathfrak{C} = \mathfrak{C}_a \neq \varnothing \text{ and } \mathfrak{C}_b = \varnothing \\ q_{\min} & \text{otherwise} \end{cases}$$

Proof. Using the argument presented in the proof of Lemma 5.3, the claim is a direct consequence of Lemma 4.33. □

Let us now consider the remaining weight quantile, i.e., the positive $\Box$-weight quantile. The characterization for the corresponding $\Box$-weight decision problem (cf. Lemma 4.35) does not yield a computation scheme for the positive $\Box$-weight quantile. Let us argue why. Notice that in contrast to $dist_{\min}$, $dist_{\max}$ and μ, *cdist* takes the given threshold z into account. Thus, trying to use *cdist* to compute the positive $\Box$-weight quantile, would resemble to a dog chasing its tail.

However, by Lemma 4.35 there exists a $z \in \mathbb{Z}$ such that $Pr_\iota^{\mathcal{M}}(\Box\mathsf{weight} > z) > 0$ if and only if there exists a BSCC of type (a) or (b). Thus, we can decide in polynomial time whether the positive $\Box$-weight evaluates to $-\infty$. If this is not the case, we can exploit that the integers are not dense in $\mathbb{R}$ and compute the quantile by deciding the positive $\Box$-weight problem for decreasing integers starting with the maximal value of weight.

Although the computation terminates and does the job, we are using a sledgehammer to crack a nut. In fact, we can amend this idea to show that the positive $\Box$-weight quantile can be computed in polynomial time, by narrowing down the interval and using binary search instead of linear search.

Let us first assume that weight is a reward function. Then the quantile has to be contained in $[0, max_{\mathsf{weight}}]$ where max_{weight} denotes the maximum value of the function weight. As the interval is exponential in the binary encoding size of weight, we can compute the positive $\Box$-reward quantile in polynomial time using a binary search and Lemma 4.35.

In fact, we can state a similar interval for weight functions, which include negative values.

Lemma 5.5. *Let* min_{weight} *and* max_{weight} *be the minimum and maximum values of the function* weight*, respectively. Assume* $min_{\mathsf{weight}} < 0$.

$$\mathsf{Qu}^{>0}[\Box\mathsf{weight}] \in ([2 \cdot |S| \cdot min_{\mathsf{weight}}, max_{\mathsf{weight}}] \cap \mathbb{Z}) \cup \{-\infty\}$$

Proof. As trivially $\mathsf{Qu}^{>0}[\Box\mathsf{weight}] < max_{\mathsf{weight}}$, it suffices to show that for every $z \in \mathbb{Z}$

$$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z)) > 0 \text{ implies } Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > 2 \cdot |S| \cdot min_{\mathsf{weight}})) > 0.$$

With the help of Lemma 4.35 there exists a BSCC $\mathcal{C}$ and $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ such that $\mathsf{last}(\hat{\pi}) \in \mathcal{C}$ and one of the following conditions holds:

(a) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ and $\hat{\pi}$ contains a positive cycle.

(b) $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and $\mathcal{C}$ does not contain a negative cycle.

In case (a) we can safely assume that $\hat{\pi}$ contains exactly one simple cycle and that this cycle is positive. Thus, one can iterate this positive simple cycle to construct a finite path $\hat{\pi}' \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ such that $\mathsf{last}(\hat{\pi}') = \mathsf{last}(\hat{\pi}) \in \mathcal{C}$ and

$$\mathsf{minwgt}(\hat{\pi}') \geq |S| \cdot min_{\mathsf{weight}} > 2 \cdot |S| \cdot min_{\mathsf{weight}}.$$

Turning to case (b), let $\hat{\pi}'' \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ be the path that results from $\hat{\pi}$ by removing all cycles. Hence $\mathsf{weight}(\hat{\pi}'') \geq |S| \cdot min_{\mathsf{weight}}$. As $\mathcal{C}$ contains only cycles with weight zero (cf. Lemma 4.14) we have $\mu(s) < -|S| \cdot min_{\mathsf{weight}}$ for all $s \in \mathcal{C}$. Thus, $\mathsf{weight}(\hat{\pi}'') > \mu(\mathsf{last}(\hat{\pi}'')) + 2 \cdot |S| \cdot min_{\mathsf{weight}}$.

This completes the proof as for both cases (a) and (b) Lemma 4.35 yields

$$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > 2 \cdot |S| \cdot min_{\mathsf{weight}})) > 0.$$

$\Box$

As discussed above, using Lemma 5.5 we can conclude the following theorem.

Theorem 5.6. *Let* $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ *be an energy-utility Markov chain and* $\phi = (2^{AP})^\omega$. *For* $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ *the qualitative* $\heartsuit$*-weight quantiles are computable in time polynomial in the size of* $\mathcal{M}$.

5.2 Ratio quantiles

The aim of this section is to show that qualitative ratio quantiles are computable in polynomial time. To do so, in Section 5.2.1 we will introduce the best-approximation problem as stated in [17] and state a reduction from qualitative ratio quantiles to this problem.

However, let us first discuss why, in contrast to the decision problems considered in the previous chapter, we cannot simply transfer the previous sections' results for qualitative weight quantiles to qualitative ratio quantiles.

Why ratio and weight quantiles are different

In contrast to the decision problems, the standard reduction of ratio constraints to weight constraints stated in Lemma 4.1 does not apply to quantiles. As this transformation depends on the threshold q, applying it for ratio quantiles would result in a dog chasing its tail.

Furthermore, $\mathbb{Q}$ is dense in $\mathbb{R}$, which has several consequences. First, stating an upper and lower bound for the quantile and applying Corollary 4.42 does not imply that qualitative ratio quantiles are computable, but only that we can approximate qualitative ratio quantiles up to an arbitrary small error $\varepsilon \in \mathbb{Q}_{>0}$. Additionally, the definition of a quantile as the supremum and not the maximum allows for the case that for some $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ and $* \in \{>0, =1\}$ we have $q = \mathsf{Qu}^*[\heartsuit\mathsf{ratio}]$ but $Pr_\iota^\mathcal{M}(\heartsuit\mathsf{ratio} > q) > 0$, i.e., the quantile is a real supremum and does not match with the maximum.

Due to these fundamental differences, we can not simply transfer the results of the previous section to ratio quantiles. Further considering dualities for ratio quantiles is not very promising. Both, the proof of Lemma 4.2 and the proof of Lemma 5.1, rely on the equivalence of $(\mathsf{weight} \geq z)$ and $(\mathsf{weight} > z - 1)$. Such an equivalence cannot be established for rational numbers.

5.2.1 A reduction to the best-approximation problem

The aim of this section is to prove that qualitative ratio quantiles can be computed in polynomial time. For now, let us concentrate on pure computability and postpone the discussion of complexity. We sketch the principle concept for the almost-sure $\Box$-ratio quantile. The argument for any other qualitative ratio quantile is based on the same idea.

Assume we could state a finite set containing the value of the almost-sure $\Box$-ratio quantile. Then, one method to actually compute the quantile would be to decide the corresponding almost-sure $\Box$-ratio decision problem for every element of this set, as the exact quantile value can be distinguished from the other elements of the set in the following way. If it has a predecessor (w.r.t the given finite set), the decision procedure has a positive result for this predecessor. For its successor (if it exists) the decision procedure has a negative result. The monotonicity of the $\Box$-ratio decision problem w.r.t. the threshold q ensures there can only be one element which satisfies both.

This approach relies on the computation of all values in the set. However, we can only state exponentially large sets. In the following we will state a polynomial computation scheme, based on a reduction to the following best-approximation problem.

The best approximation problem

We use the best-approximation problem as defined by Grötschel, Lovász, and Schrijver in [17].

Definition 5.7 ([17] Problem 5.1.1). *Given a natural number $N \in \mathbb{N}_{>0}$ and a rational number $\alpha \in \mathbb{Q}$, the* best-approximation problem for N and α *asks for some $\beta \in \mathbb{Q}$ with denominator at most N that minimizes $|\alpha - \beta|$.*

Example 5.8. Assume $N = 10$. If $\alpha = 1/3$ the solution to the best-approximation problem is $1/3$ as the dominator of α is less or equal to N. However, if $\alpha = 10/33$ the solution is $3/10$.

Using the continued-fraction method, one can show that for given N and α the best-approximation problem is solvable in polynomial time. We will not restate the proof, but consider the following theorem in the remainder of this section as a black box.

Theorem 5.9 ([17], Theorem 5.1.9). *Let $N \in \mathbb{N}$, $\alpha \in \mathbb{Q}$. The best-approximation problem for $N \in \mathbb{N}_{>0}$ and $\alpha \in \mathbb{Q}$ is solvable in time polynomial in the encoding length of N and α.*

Let us now sketch how the best-approximation problem relates to ratio quantiles. Above we discussed the idea to state a finite set and check for every element, whether it constitutes the quantile value or not. Now, assume we only know that such a set exists and can compute the largest denominator of all elements. In the following we will denote this value by N. Further, assume that we can approximate the quantile up to an high arbitrary precision $1/\varepsilon$ and let α be such an approximation of the almost-sure $\square$-ratio quantile. If this ε is *small enough*, i.e., α is closer to the actual quantile than to any other value with denominator at most N, the quantile can be computed using the best-approximation problem for N and α. The following lemma formalizes this argument and characterizes *small enough*.

Lemma 5.10. *For $N \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$ there exists at most one $\beta \in \mathbb{Q}$ such that*

(1) $|\alpha - \beta| < 1/(2N^2)$*, and*

(2) $\beta \in \{a/b : a \in \mathbb{N} \text{ and } b \in [1, N] \cup \mathbb{N}\}$

If such β exists, it is computable in time polynomial in the size of N and α w.r.t. their encoding length.

Proof. Let $\gamma \in \mathbb{Q}$ be the solution of the best-approximation problem for N and α. Let β be a rational number satisfying (1) and (2). We now argue that $\gamma = \beta$. A similar argument has already been presented in [14].

By definition of the best-approximation problem and assumption (2), we obtain $|\alpha - \gamma| \leq |\alpha - \beta|$, as both γ and β are contained in $\{a/b : a \in \mathbb{N} \text{ and } b \in [1, N] \cap \mathbb{N}\}$ and γ minimizes $|\alpha - \gamma|$. Using the triangle inequality, this yields,

$$|\gamma - \beta| \leq |\gamma - \alpha| + |\alpha - \beta| \leq 2 \cdot |\alpha - \beta| < \frac{1}{N^2}. \tag{5.1}$$

Towards a contradiction, assume that $\gamma \neq \beta$. Let $\gamma = n_1/n_2$ and $\beta = m_1/m_2$ for some natural numbers n_1, n_2, m_1 and m_2. Since both $n_2 \leq N$ and $m_2 \leq N$, and the assumption $\gamma \neq \beta$ implies $m_1n_2 - n_1m_2 \neq 0$, we obtain

$$|\gamma - \beta| = \left| \frac{m_1n_2 - n_1m_2}{n_2m_2} \right| \geq \frac{|m_1n_2 - n_1m_2|}{N^2} \geq \frac{1}{N^2}$$

This contradicts Equation (5.1) and therefore yields $\gamma = \beta$. Thus, γ fulfills the requirements (1) and (2). This completes the proof, as γ is the solution of the best-approximation problem for N and α which by Theorem 5.9 is computable in time polynomial in the encoding size of N and α. □

Considering the above lemma it seems natural to investigate the computational complexity of approximating qualitative ratio quantiles and computing an upper bound for the denominator of the actual value. In the remainder of this section we will show that both can be done in polynomial time, i.e., the following two statements hold:

(I) Given $N \in \mathbb{N}$ we can approximate qualitative ratio quantiles up to precision $1/(2N^2)$ in polynomial time.

(II) We can compute an upper bound for the denominator of any qualitative ratio quantile in polynomial time.

Before we prove these two claims, let us discuss why the above statements (I) and (II) imply that qualitative ratio quantiles can be computed in polynomial time. Using Lemma 5.10, we can state the following polynomial-time computation procedure for qualitative ratio quantiles.

- Compute the upper bound N for the denominator.
- Approximate the quantile up to precision $1/(2N^2)$. This yields value α.
- Solve the best-approximation problem for N and α, the solution equals the value of the considered quantile.

Let us now consider proofs for the above two claims (I) and (II).

Approximating ratio quantiles

As already mentioned at the beginning of the section, the results of Chapter 4 imply that stating an upper and lower bound for qualitative ratio quantiles suffices to prove claim (I) .

Lemma 5.11. *Given $N \in \mathbb{N}$, $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$, the quantile* $\mathsf{Qu}^*[\heartsuit\mathsf{ratio}]$ *can be approximated up to precision* $1/(2N^2)$ *in time polynomial in the encoding length of N and the size of $\mathcal{M}$.*

Proof. The qualitative $\heartsuit$-ratio problems are decidable in polynomial time (Corollary 4.42). Hence, given $r \in \mathbb{Q}$ such that $\mathsf{Qu}^*[\heartsuit\mathsf{ratio}] \leq r$, we can approximate the quantile up to an arbitrary precision ε in time polynomial in the size of $\mathcal{M}$ and logarithmic in $1/\varepsilon$ and r using a binary search on the interval $[0, r]$.

Let $T \subseteq S \times S$ be the set containing all pairs of states (s, s') such that $P(s, s') > 0$. To complete the proof it suffices to show

$$\mathsf{Qu}^*[\heartsuit\mathsf{ratio}] \leq \frac{\max_{(s,s') \in T} \mathsf{utility}(s, s')}{\min_{(s,s') \in T} \mathsf{energy}(s, s')}, \tag{5.2}$$

as both the minimum and the maximum are computable in time polynomial in the size of energy and $\mathsf{utility}$. Notice, that the fact $(a+b)/(c+d) \leq \max(a,b)/\min(c,d)$ implies for every finite path $\hat{\pi}$

$$\mathsf{ratio}(\hat{\pi}) \leq \frac{\max_{(s,s') \in T} \mathsf{utility}(s, s')}{\min_{(s,s') \in T} \mathsf{energy}(s, s')},$$

which directly yields Equation (5.2) and therefore completes the proof. □

Bounding the denominator

The argument to prove the second claim, i.e., that we can compute an upper bound for the denominator is more involved. We devote the rest of the section to the proof of the following lemma.

Lemma 5.12. *One can compute a natural number N in polynomial time such that*

$$\mathsf{Qu}^*[\heartsuit\mathsf{ratio}] \in \{a/b : a \in \mathbb{N} \text{ and } b \in [1, N] \cap \mathbb{N}\}.$$

As mentioned before, the idea is to state a finite set of rational numbers which serve as candidates for the exact quantile values and then compute an upper bound for the denominators of the elements in this set. In the following, we will show that $Q_{all} = Q_{bscc} \cup Q_{path} \cup Q_{cycle}$, where

$$\begin{aligned} Q_{bscc} &= \{\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) : \ \mathcal{C} \text{ is a BSCC of } \mathcal{M}\}, \\ Q_{path} &= \{\mathsf{ratio}(\hat{\pi}) : \ \hat{\pi} \text{ is a simple path starting in } \iota\}, \text{ and} \\ Q_{cycle} &= \{\mathsf{ratio}(\vartheta) : \ \vartheta \text{ is a simple cycle}\} \end{aligned}$$

fulfills all the requirements for such a set.

Let us first discuss that we can compute an upper bound for the denominators in Q_{all} in polynomial time. For both, Q_{path} and Q_{cycle}, this trivially is the case, as we are only considering finite paths, i.e., of length at most $|S|$. Thus, one upper bound is $|S| \cdot \max_{(s,s') \in S^2} \mathsf{energy}(s, s')$. Note that, for Markov chains both, the BSCCs and their expected long-run ratios, are computable in polynomial time. This implies that Q_{bscc} is computable in polynomial time and hence Lemma 5.12 is a direct consequence of the following lemma.

Lemma 5.13.

$$\mathsf{Qu}^*[\heartsuit \mathsf{ratio}] \in Q_{all}$$

Before we get to the proof, let us investigate the set Q_{all}.

Example 5.14. Reconsider the energy-utility Markov chain of Example 3.6 depicted in Figure 3.2. The Markov chain has exactly one BSCC $\mathcal{C} = \{c_1, c_2, c_3\}$, six simple paths starting in ι and four simple cycles.

$$\begin{aligned}
\text{simple paths: } & \iota t_1,\ \iota t_1 t_2,\ \iota t_1 c_1,\ \iota t_1 c_1 c_2,\ \iota t_1 c_1 c_2 c_3,\ \iota t_1 c_1 c_3 \\
\text{simple cycles: } & t_1 t_2 t_1,\ c_1 c_2 c_3 c_1,\ c_1 c_3 c_1,\ c_3 c_3
\end{aligned}$$

Thus, dependent on the chosen variant we get the following sets (remember that $\mathsf{energy}_{v_1}(t_1, c_1) = 2$, $\mathsf{energy}_{v_2}(t_1, c_1) = 4$ and $\mathsf{energy}_{v_3}(t_1, c_1) = 8$):

$$\begin{array}{llll}
v_1: & Q_{bscc} = \left\{\frac{6}{7}\right\} & Q_{path} = \left\{\frac{1}{1}, \frac{3}{5}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}\right\} & Q_{cycle} = \left\{\frac{4}{8}, \frac{4}{3}, \frac{2}{2}, \frac{0}{1}\right\} \\
v_2: & Q_{bscc} = \left\{\frac{6}{7}\right\} & Q_{path} = \left\{\frac{1}{1}, \frac{3}{5}, \frac{4}{5}, \frac{6}{6}, \frac{8}{7}\right\} & Q_{cycle} = \left\{\frac{4}{8}, \frac{4}{3}, \frac{2}{2}, \frac{0}{1}\right\} \\
v_3: & Q_{bscc} = \left\{\frac{6}{7}\right\} & Q_{path} = \left\{\frac{1}{1}, \frac{3}{5}, \frac{4}{9}, \frac{6}{10}, \frac{8}{11}\right\} & Q_{cycle} = \left\{\frac{4}{8}, \frac{4}{3}, \frac{2}{2}, \frac{0}{1}\right\}
\end{array}$$

Using the same energy-utility Markov chain we can also show that each of the set Q_{bscc}, Q_{path} and Q_{cycle} may provide candidates for the quantile values. As mentioned before the Markov chain contains exactly one BSCC $\mathcal{C} = \{c_1, c_2, c_3\}$ with $\mathbb{L}_{\mathcal{C}}(ratio) = 6/7$. Thus $\mathsf{Qu}^{>0}[\Box \mathsf{ratio}] \leq 6/7$.

Using Lemma 4.35 and Lemma 4.1, we can argue that for variant v_1 the positive $\Box$-ratio quantile in fact evaluates to $6/7$. Consider the path $\hat{\pi} = \iota t_1 c_1 c_3 c_1$. For any prefix ρ of $\hat{\pi}$ we have $\mathsf{ratio}(\rho) > 6/7$. Thus, for arbitrary $\varepsilon \in \mathbb{Q}_{>0}$ the path $\hat{\pi}$ contains a positive cycle w.r.t. $\mathsf{weight}_{(6/7)-\varepsilon}$ and $\mathsf{minwgt}[\mathsf{weight}_{(6/7)-\varepsilon}](\hat{\pi}) > 0$. Further, $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}_{(6/7)-\varepsilon}) > 0$. Thus, $\mathsf{Qu}^{>0}[\Box \mathsf{ratio}] = 6/7$ as $Pr_{\iota}^{\mathcal{M}}(\Box \mathsf{ratio} > (6/7) - \varepsilon) > 0$ but $Pr_{\iota}^{\mathcal{M}}(\Box \mathsf{ratio} > 6/7) = 0$. Notice that $6/7$ is contained in Q_{bscc} but neither in Q_{path} nor Q_{cycle} of variant v_1.

Let us now consider variant v_2. Using a similar argument, we can state that $\mathsf{Qu}^{>0}[\Box \mathsf{ratio}] = 4/5$, as every path $\hat{\pi}$ from ι to c_1 has to contain a prefix ρ (including $\rho = \hat{\pi}$) with $\mathsf{ratio}(\rho) \leq 4/5$. Notice that for variant v_2 $4/5$ is only contained in Q_{path}.

For variant v_3 the positive $\Box$-ratio quantile evaluates to $1/2$ which is contained in Q_{cycle} but not in $Q_{bscc} \cup Q_{path}$. Let us see why. We have

$$\lim_{n\to\infty} \frac{1+4\cdot n+3}{1+8\cdot n+8} = \frac{1}{2}.$$

Thus, for every $q < 1/2$ there exists a finite path starting in ι and ending in $\mathcal{C}$ such that for every prefix ρ of $\hat{\pi}$ holds $\mathsf{ratio}(\rho) > q$. Hence using the arguments from above we have $Pr_\iota^{\mathcal{M}}(\Box\mathsf{ratio} > q) > 0$ but $Pr_\iota^{\mathcal{M}}(\Box\mathsf{ratio} > 1/2) = 0$ which yields $\mathsf{Qu}^{>0}[\Box\mathsf{ratio}] = 1/2$.

Let us now return to Lemma 5.13. To prove the claim we will consider the four modalities $\{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ individually. For the long-run modalities $\{\Diamond\Box, \Box\Diamond\}$ the proofs are based on the fact that for almost all paths reaching a BSCC $\mathcal{C}$, the ratio of their prefixes converges towards the long-run ratio $\mathbb{L}_{\mathcal{C}}(ratio)$ of this BSCC.

Lemma 5.15.

$$\mathsf{Qu}^{=1}[\Box\Diamond\mathsf{ratio}] = \mathsf{Qu}^{=1}[\Diamond\Box\mathsf{ratio}] = \min Q_{bscc} \textit{ and}$$
$$\mathsf{Qu}^{>0}[\Box\Diamond\mathsf{ratio}] = \mathsf{Qu}^{>0}[\Diamond\Box\mathsf{ratio}] = \max Q_{bscc}$$

Proof. Let $r \in \mathbb{Q}$ and abbreviate $\min Q_{bscc}$ by $q_{\min}$ and $\max Q_{bscc}$ by $q_{\max}$. As

$$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{ratio} > r)) \le Pr_\iota^{\mathcal{M}}(\Box\Diamond(\mathsf{ratio} > r))$$

it suffices to prove the following four implications.

(a) $Pr_\iota^{\mathcal{M}}(\Box\Diamond(\mathsf{ratio} > r)) > 0$ implies $r \le q_{\max}$

(b) $r < q_{\max}$ implies $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{ratio} > r)) > 0$

(c) $Pr_\iota^{\mathcal{M}}(\Box\Diamond(\mathsf{ratio} > r)) = 1$ implies $r \le q_{\min}$

(d) $r < q_{\min}$ implies $Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{ratio} > r)) = 1$

Proof of (a). Remind that $Pr_\iota^{\mathcal{M}}$-almost all infinite paths π eventually reach a BSCC $\mathcal{C}$ and $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = \lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n])$. Thus, $Pr_\iota^{\mathcal{M}}$-almost all infinite paths π satisfy

$$q_{\min} \le \lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n]) \le q_{\max}. \tag{5.3}$$

Proof of (b). Assume $r < q_{\max}$ and let $\mathcal{C}$ be a BSCC with $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = q_{\max}$. For $Pr_\iota^{\mathcal{M}}$-almost all paths π eventually reaching $\mathcal{C}$ we have

$$q_{\max} = \mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = \lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n]).$$

Thus for every $\varepsilon \in \mathbb{Q}_{>0}$ and for almost all paths eventually reaching $\mathcal{C}$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\ge m}$ $\mathsf{ratio}(\pi[\ldots n]) \ge q_{\max} - \varepsilon$. Hence,

$$Pr_\iota^{\mathcal{M}}(\Diamond\Box(\mathsf{ratio} > r)) \ge Pr_\iota^{\mathcal{M}}(\Diamond\mathcal{C}) > 0.$$

Proof of (c). We show this by contraposition. Assume $r > q_{\min}$ and let $\mathcal{C}$ be a BSCC with $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = q_{\min}$. For all $\varepsilon \in \mathbb{Q}_{>0}$ and almost all paths π reaching $\mathcal{C}$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq m}$ we have $\lim_{n\to\infty} \mathsf{ratio}(\pi[\ldots n]) < q_{\min} + \varepsilon$. Hence,

$$Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > q) \leq 1 - Pr_\iota^{\mathcal{M}}(\Diamond\mathcal{C}) < 1.$$

Proof of (d). Implication (d) follows directly from inequality (5.3), as thus for $Pr_\iota^{\mathcal{M}}$-almost all infinite path $\mathsf{InfPaths}$ and $\varepsilon \in \mathbb{R}_{>0}$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq m}$

$$\mathsf{ratio}(\pi[\ldots n]) \geq q_{\min} - \varepsilon.$$

□

In fact, as Q_{bscc} is computable in polynomial time, Lemma 5.15 directly yields that qualitative $\heartsuit$-ratio quantiles for $\heartsuit \in \{\Diamond\Box, \Box\Diamond\}$ are computable in polynomial time.

The proofs for $\Box$ and $\Diamond$ are more involved. In the following we will first consider the qualitative $\Box$-ratio quantiles and then state analogous proofs for the qualitative $\Diamond$-ratio quantiles.

Lemma 5.16.

$$\mathsf{Qu}^{=1}[\Box\mathsf{ratio}] = \min(Q_{path} \cup Q_{cycle})$$

Proof. The claim follows if the following two statements are true. Given $r \in \mathbb{Q}$,

(a) $Pr_\iota^{\mathcal{M}}(\mathsf{ratio} > r) = 1$ implies $r \leq q_{\min}$ and,

(b) $r < q_{\min}$ implies $Pr_\iota^{\mathcal{M}}(\mathsf{ratio} > r) = 1$.

Proof of (a). To prove the first statement, we show that for every $r' \in Q_{path} \cup Q_{cycle}$ and every $\varepsilon \in \mathbb{R}_{>0}$ there is a finite path $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ such that $\mathsf{ratio}(\hat{\pi}) \leq r' + \varepsilon$. This trivially is the case if $r' \in Q_{path}$. Assume $r' \in Q_{cycle} \setminus Q_{path}$. Let ϑ be a simple cycle with $\mathsf{ratio}(\vartheta) = r'$. Given a finite path $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ reaching $\mathsf{first}(\vartheta)$, i.e., such that $\hat{\pi} \diamond \vartheta \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$, there exists $k \in \mathbb{N}$ such that $\hat{\pi} = \hat{\pi} \diamond (\vartheta)^k$ satisfies $\mathsf{ratio}(\hat{\pi}) \leq r' + \varepsilon$ (as before $(\vartheta)^i$ is defined as $\vartheta^0 = \mathsf{first}(\vartheta)$ and $\vartheta \diamond (\vartheta)^{i-1}$). This follows from the fact that for all natural numbers a, b, c, and d with $c \neq 0$ and $d \neq 0$ we have $\lim_{n\to\infty}(a + b \cdot n)/(c + d \cdot n) = b/d$.

Proof of (b). Towards a contradiction, assume that there are $\varepsilon \in \mathbb{R}_{>0}$ and $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ such that $\mathsf{ratio}(\hat{\pi}) = q_{\min} - \varepsilon$. Let Θ be the set of all simple cycles occurring in $\hat{\pi}$ and denote by $\hat{\pi}'$ the simple path that results from $\hat{\pi}$ by iteratively removing all simple cycles. We have

$$\mathsf{ratio}(\hat{\pi}) \geq \min\big(\{\mathsf{ratio}(\hat{\pi}')\} \cup \{\mathsf{ratio}(\vartheta) \colon \vartheta \in \Theta\}\big) \geq q_{\min},$$

as for all natural numbers a, b, c, and d with $c \neq 0$ and $d \neq 0$ holds $(a+b)/(c+d) \geq \min\{a/c,\, b/d\}$. But this contradicts $\mathsf{ratio}(\hat{\pi}) = q_{\min} - \varepsilon$. □

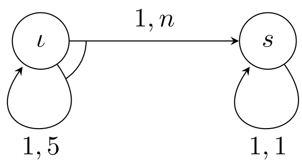

Figure 5.1: Energy-utility Markov chain (transitions are labeled with tuples (utility, energy) and choices are assumed to be uniformly distributed)

Let us sketch the key principle of the proof for $\mathsf{Qu}^{>0}[\Box\mathsf{ratio}] \in Q_{all}$. We show that $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > q)) > 0$ for some $q \notin Q_{all}$ implies that there exists a $q' \in Q_{all}$, $q < q'$ such that $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > q' - \varepsilon)) > 0$ for all $\varepsilon \in \mathbb{Q}_{>0}$. Using the assumption $q \notin Q_{all}$, Lemma 4.1 and Lemma 4.35 yield that there exists a path $\hat{\pi}_q$ reaching a BSCC $\mathcal{C}$ such that

- $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) > q$,
- $\hat{\pi}_q$ contains a cycle ϑ with $\mathsf{ratio}(\vartheta) > q$ and
- $\mathsf{ratio}(\rho) > q$ for every nonempty prefix ρ of $\hat{\pi}_q$,

which can be seen as a *witness* for $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > q)) > 0$. Let $q' \in Q_{all}$ be the smallest element of Q_{all} such that $q < q'$. Based on $\hat{\pi}_q$, for every $\varepsilon \in \mathbb{Q}_{>0}$ we can construct a finite path $\hat{\pi}_{q'-\varepsilon} \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota\ldots\mathsf{last}(\hat{\pi}_q)}$, which serves as a witness for $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > q' - \varepsilon)) > 0$. The construction relies on the following observation. For any finite path $\hat{\pi}$ there exists a finite path $\hat{\pi}' = \hat{\pi}_0 \diamond (\vartheta)^k \diamond \hat{\pi}_1$, where $\hat{\pi}_0$, $\hat{\pi}_1$ are simple paths and ϑ is a simple cycle such that $\mathsf{first}(\hat{\pi}) = \mathsf{first}(\hat{\pi}')$, $\mathsf{last}(\hat{\pi}) = \mathsf{last}(\hat{\pi}')$ and $\hat{\pi}'$ is *better* than $\hat{\pi}$ wrt. ratio. In this context *better* means $\mathsf{ratio}(\hat{\pi}') \geq \mathsf{ratio}(\hat{\pi})$ and

$$\min_{n\in[1,|\hat{\pi}'|]} \mathsf{ratio}(\hat{\pi}'[\ldots n]) \geq \min_{n\in[1,|\hat{\pi}|]} \mathsf{ratio}(\hat{\pi}[\ldots n]).$$

Example 5.17. Consider the Markov chain depicted in Figure 5.1. Assume $n = 100$. Using Lemma 4.1 and Lemma 4.35, the finite path $\hat{\pi}_{3/100} = \iota\iota s$ is a witness for $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > 3/100)) > 0$. Since $\iota\iota$ is a simple cycle and $\mathsf{ratio}(\iota\iota) = 1/5$, given $\varepsilon \in \mathbb{Q}_{>0}$ there exists $k \in \mathbb{N}$ such that $\hat{\pi}_{1/5-\varepsilon} = \iota \diamond (\iota\iota)^k \diamond \iota s$ is a witness for $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > 1/5 - \varepsilon)) > 0$.

If we assume $n = 1$, then $\hat{\pi}_{1/6} = \iota\iota s$ is a witness for $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > 1/6)) > 0$. As $\mathsf{ratio}(\iota\iota) = 1/5 < \mathsf{ratio}(\hat{\pi}_{1/6})$ we can ommit the cycle $\iota\iota$. For every $\varepsilon \in \mathbb{Q}_{>0}$ the path $\hat{\pi}_{1-\varepsilon} = \iota s$ is a witness for $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{ratio} > 1 - \varepsilon)) > 0$.

Lemma 5.18.

$$\mathsf{Qu}^{*}[\Box\mathsf{ratio}] \in Q_{all}$$

Proof. By Lemma 5.16 we have $\mathsf{Qu}^{=1}[\Box\mathsf{ratio}] \in Q_{all}$. To show $\mathsf{Qu}^{>0}[\Box\mathsf{ratio}] \in Q_{all}$, we prove the following claim.

Let $a, b \in Q_{all}$ be such that $a < b$ and there is no $m \in Q_{all}$ satisfying $a < m < b$. Let $c \in \mathbb{Q}$ such that $a < c < b$. If $Pr_\iota^\mathcal{M}(\Box(\mathsf{ratio} > c)) > 0$ then for every $\varepsilon \in \mathbb{Q}_{>0}$ $Pr_\iota^\mathcal{M}(\Box(\mathsf{ratio} > b - \varepsilon)) > 0$.

As before for q_1, $q_2 \in \mathbb{Z}$ let $\mathsf{weight}_{q_1/q_2} \colon S \times S \to \mathbb{Z}$ be given by

$$\mathsf{weight}_{q_1/q_2} = q_2 \cdot \mathsf{utility} - q_1 \cdot \mathsf{energy}.$$

Assume that $Pr_\iota^\mathcal{M}(\Box(\mathsf{ratio} > c)) > 0$. Hence we have $Pr_\iota^\mathcal{M}(\Box(\mathsf{weight}_c > 0)) > 0$ (cf. Lemma 4.1). Let c_1, $c_2 \in \mathbb{Z}$ be such that $c = c_1/c_2$. Notice that $\mathcal{M}$ cannot contain a BSCC $\mathcal{C}$ with $\mathbb{E}_\mathcal{C}(\mathsf{weight}_c) = 0$. This would imply $\mathbb{L}_\mathcal{C}(\mathsf{ratio}) = c$ and thus yield $c \in Q_{bscc}$, which contradicts the assumption that there is no $m \in Q_{all}$ such that $a < m < b$. Hence by Lemma 4.35 there exists a finite path $\hat{\pi} \in \mathsf{FinPaths}_\iota^\mathcal{M}$ and a BSCC $\mathcal{C}$ such that $\mathbb{E}_\mathcal{C}(\mathsf{weight}_c) > 0$, $\mathsf{last}(\hat{\pi}) \in \mathcal{C}$, $\mathsf{minwgt}[\mathsf{weight}_c](\hat{\pi}) > 0$ and $\hat{\pi}$ contains a weight_c-positive cycle. We can safely assume that all simple cycles of $\hat{\pi}$ are weight_c-positive.

We now use $\mathcal{C}$ and $\hat{\pi}$ to show that $Pr_\iota^\mathcal{M}(\Box(\mathsf{ratio} > b - \varepsilon)) > 0$ for every $\varepsilon \in \mathbb{Q}_{>0}$. Let us fix an $\varepsilon \in \mathbb{Q}_{>0}$. To be able to apply Lemma 4.35 for $\mathsf{weight}_{b-\varepsilon}$, we need to show that $\mathbb{E}_\mathcal{C}(\mathsf{weight}_{b-\varepsilon}) > 0$ and find a path $\hat{\pi}'$ reaching $\mathcal{C}$ satisfying $\mathsf{minwgt}[\mathsf{weight}_{b-\varepsilon}](\hat{\pi}) > 0$, which contains a $\mathsf{weight}_{b-\varepsilon}$-positive cycle. $\mathbb{E}_\mathcal{C}(\mathsf{weight}_{b-\varepsilon}) > 0$ follows directly from the assumption that there is no $m \in Q_{all}$ such that $a < m < b$ and $\mathbb{E}_\mathcal{C}(\mathsf{weight}_c) > 0$, as

$$\mathbb{L}_\mathcal{C}(\mathsf{ratio}) > c > a \quad \text{implies} \quad \mathbb{L}_\mathcal{C}(\mathsf{ratio}) \geq b.$$

Let us now investigate $\hat{\pi}$ to prove the existence of the required $\hat{\pi}'$. We denote the left-most simple cycle of $\hat{\pi}$ by ϑ. The requirements on $\hat{\pi}$ ensure that ϑ is a weight_c-positive cycle. Let $\hat{\pi}_0, \hat{\pi}_1 \in \mathsf{FinPaths}_\iota^\mathcal{M}$ such that $\hat{\pi} = \hat{\pi}_0 \diamond \vartheta \diamond \hat{\pi}_1$ and ϑ is not contained in $\hat{\pi}_0$. The finite path $\hat{\pi}_0 \diamond \vartheta$ contains exactly one (simple) cycle, namely ϑ. Therefore, $\mathsf{ratio}(\rho) \in Q_{path}$ for every real prefix ρ of $\hat{\pi}_0 \diamond \vartheta$, i.e., $\rho \neq \hat{\pi}_0 \diamond \vartheta$ and $|\rho| > 0$. Furthermore, $\mathsf{ratio}(\rho) > c$ for every prefix of $\hat{\pi}$ as $\mathsf{minwgt}[\mathsf{weight}_c](\hat{\pi}) > 0$. Now, using the same argument as for $\mathbb{L}_\mathcal{C}(\mathsf{ratio})$, we obtain $\mathsf{ratio}(\rho) \geq b$ for every real prefix of $\hat{\pi}_0 \diamond \vartheta$. A similar argument yields $\mathsf{ratio}(\vartheta) \geq b$ as ϑ is weight_c-positive cycle, i.e., $\mathsf{ratio}(\vartheta) > c > a$.

Hence, there is a $k \in \mathbb{N}$ such that $\hat{\pi}' = \hat{\pi}_0 \diamond (\vartheta)^k \diamond \hat{\pi}_1$ fulfills the requirements, i.e., $\hat{\pi}'$ contains a $\mathsf{weight}_{b-\varepsilon}$-positive cycle, namely ϑ, $\mathsf{minwgt}[\mathsf{weight}_{b-\varepsilon}](\hat{\pi}') > 0$ and $\hat{\pi}$ is a finite path starting in ι and reaching BSCC $\mathcal{C}$ such that $\mathbb{E}_\mathcal{C}(\mathsf{weight}_{b-\varepsilon}) > 0$. Thus, by Lemma 4.35 $Pr_\iota^\mathcal{M}(\Box(\mathsf{weight}_{b-\varepsilon} > 0)) > 0$. Hence, $Pr_\iota^\mathcal{M}(\Box(\mathsf{ratio} > b - \varepsilon)) > 0$. $\Box$

Lemma 5.19.

$$\mathsf{Qu}^{>0}[\Diamond \mathsf{ratio}] = \max(Q_{path} \cup Q_{cycle})$$

Proof. We show the following two implications. Given $r \in \mathbb{Q}$,

(a) $Pr_\iota^\mathcal{M}(\Diamond(\mathsf{ratio} > r)) > 0$ implies $r \leq q_{\max}$ and,

(b) $r < q_{\max}$ implies $Pr_\iota^\mathcal{M}(\Diamond(\mathsf{ratio} > r)) > 0$.

Proof of (a). The first implication is an immediate consequence of the fact that $(a+b)/(c+d) \leq \max\{a/c, b/d\}$ for all natural numbers a, b, c, and d, where $c \neq 0$ and $d \neq 0$. Given finite paths $\hat{\pi}$, $\hat{\pi}_1$ and $\hat{\pi}_2$ such that $\hat{\pi} = \hat{\pi}_1 \diamond \hat{\pi}_2$, we thus have

$$\mathsf{ratio}(\hat{\pi}) \leq \max\{\mathsf{ratio}(\hat{\pi}_1), \mathsf{ratio}(\hat{\pi}_2)\}.$$

Proof of (b). We show that for every $\varepsilon \in \mathbb{Q}_{>0}$ there exists a path $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ such that $\mathsf{ratio}(\hat{\pi}) \geq q_{\max} - \varepsilon$, which yields implication (b). If $q_{\max} \in Q_{path}$, this trivially holds. Assume $q_{\max} \in Q_{cycle} \setminus Q_{path}$ and let ϑ be a simple cycle such that $\mathsf{ratio}(\vartheta) = q_{\max}$. Further let $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ be a finite path where $\mathsf{last}(\hat{\pi}) = \mathsf{first}(\vartheta)$. Then, for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $k \in \mathbb{N}$ such that $\mathsf{ratio}(\hat{\pi} \diamond (\vartheta)^k) \geq q_{\max} - \varepsilon$. □

Lemma 5.20.

$$\mathsf{Qu}^*[\Diamond \mathsf{ratio}] \in Q_{all}$$

Proof. $\mathsf{Qu}^{>0}[\Diamond\mathsf{ratio}] \in Q_{all}$ follows directly from Lemma 5.19. In order to show $\mathsf{Qu}^{=1}[\Diamond\mathsf{ratio}] \in Q_{all}$ we prove the following two claims.

(a) $\min Q_{all} \leq \mathsf{Qu}^{=1}[\Diamond\mathsf{ratio}] \leq \max Q_{all}$

(b) Let $a, b \in Q_{all}$ such that $a < b$ and there exists no $m \in Q_{all}$ such that $a < m < b$. Further let $c \in \mathbb{Q}$ such that $a < c < b$. If $Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{ratio} > c)) < 1$, then for all $\varepsilon \in \mathbb{Q}_{>0}$, $Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{ratio} > \alpha + \varepsilon)) < 1$.

Proof of (a). Given a finite path $\hat{\pi} = \hat{\pi}_0 \diamond \hat{\pi}_1$, applying the argument presented in the proof of Lemma 5.19, we have both, $\min(\mathsf{ratio}(\hat{\pi}_0), \mathsf{ratio}(\hat{\pi}_1)) \leq \mathsf{ratio}(\hat{\pi})$ and $\mathsf{ratio}(\hat{\pi}) \leq \max(\mathsf{ratio}(\hat{\pi}_0), \mathsf{ratio}(\hat{\pi}_1))$. Hence, for all infinite paths π and all $n \in \mathbb{N}$

$$\min(Q_{path} \cup Q_{cycle}) \leq \mathsf{ratio}(\pi[\dots n]) \leq \max(Q_{path} \cup Q_{cycle}).$$

As $\min Q_{all} \leq \min(Q_{path} \cup Q_{cycle})$ and $\max Q_{all} \geq \max(Q_{path} \cup Q_{cycle})$, this completes the proof for claim (a).

Proof of (b). As usual for q_1, $q_2 \in \mathbb{Z}$ let $\mathsf{weight}_{q_1/q_2} \colon S \times S \to \mathbb{Z}$ be given by

$$\mathsf{weight}_{q_1/q_2} = q_2 \cdot \mathsf{utility} - q_1 \cdot \mathsf{energy}.$$

In order to proof the claim, we assume that $Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{ratio} > c)) < 1$ which implies $Pr_\iota^{\mathcal{M}}(\Diamond(\mathsf{weight}_c > 0)) < 1$. Hence Lemma 4.2 yields $Pr_\iota^{\mathcal{M}}(\Box(-\mathsf{weight}_c > -1)) > 0$. The basic idea of the proof is analogous to the proof of Lemma 5.18. We use Lemma 4.35 to find a witness path for $-\mathsf{weight}_c$ and then for every $\varepsilon \in \mathbb{Q}_{>0}$ build a witness path for $-\mathsf{weight}_{a+\varepsilon}$.

First of all, notice that $\mathcal{M}$ cannot contain a BSCC $\mathcal{C}$ with $\mathbb{L}_{\mathcal{C}}(\mathsf{ratio}) = c$. Thus, there is no BSCC $\mathcal{C}$ with $\mathbb{E}_{\mathcal{C}}(-\mathsf{weight}_c) = 0$ and for every BSCC $\mathcal{C}$ $\mathbb{E}_{\mathcal{C}}(-\mathsf{weight}_c) > 0$ implies $\mathbb{E}_{\mathcal{C}}(-\mathsf{weight}_{a+\varepsilon}) > 0$ for every $\varepsilon \in \mathbb{Q}_{>0}$.

Hence, Lemma 4.35 yields the existence of a path $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ and a BSCC $\mathcal{C}$ such that $\mathsf{last}(\hat{\pi}) \in \mathcal{C}$, $\mathsf{minwgt}[-\mathsf{weight}_{\mathcal{C}}](\hat{\pi}) > -1$, $\hat{\pi}$ contains a $(-\mathsf{weight}_c)$-positive cycle and $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}_{a+\varepsilon}) > 0$ for every $\varepsilon \in \mathbb{Q}_{>0}$. Let us fix an $\varepsilon \in \mathbb{Q}_{>0}$. We now use $\hat{\pi}$ to construct a witness path $\hat{\pi}'$ for $(-\mathsf{weight}_{a+\varepsilon})$, by repeating the arguments of the proof of Lemma 5.18.

We can safely assume that all simple cycles of $\hat{\pi}$ are $(-\mathsf{weight}_c)$-positive. Let ϑ be the first simple cycle in $\hat{\pi}$, i.e., ϑ is $(-\mathsf{weight}_c)$-positive. Let $\hat{\pi}_0, \hat{\pi}_1 \in \mathsf{FinPaths}^{\mathcal{M}}$ be finite paths such that $\hat{\pi} = \hat{\pi}_0 \diamond \vartheta \diamond \hat{\pi}_1$ and $\hat{\pi}_0$ does not contain ϑ.

By the choice of $\hat{\pi}_0$ and ϑ, $\mathsf{ratio}(\rho) \in Q_{path}$ for every real prefix ρ of $\hat{\pi} \diamond \vartheta$, i.e., $\rho \neq \hat{\pi} \diamond \vartheta$ and $|\rho| > 0$. As $\mathsf{minwgt}[-\mathsf{weight}_c](\hat{\pi}) > -1$ and there is no $m \in Q_{all}$ with $a < m < b$, we obtain for every real prefix ρ of $\hat{\pi}_0 \diamond \vartheta$, $(-\mathsf{weight}_c)(\rho) \geq 0$ and thus $\mathsf{ratio}(\rho) \leq c$ which implies $\mathsf{ratio}(\rho) \leq a$. Similar arguments and the fact that ϑ is $(-\mathsf{weight}_c)$-positive yield $\mathsf{ratio}(\vartheta) \leq a$.

Putting things together, there exists a $k \in \mathbb{N}$ such that $\hat{\pi}' = \hat{\pi}_0 \diamond (\vartheta)^k \diamond \hat{\pi}_1$ fulfills the requirements, i.e., $\hat{\pi}'$ contains a $(-\mathsf{weight}_{a+\varepsilon})$-positive cycle (namely ϑ), $\mathsf{minwgt}[-\mathsf{weight}_{a+\varepsilon}](\hat{\pi}') > 0$ and $\mathsf{last}(\hat{\pi}') \in \mathcal{C}$ with $\mathbb{E}_{\mathcal{C}}(-\mathsf{weight}_{a+\varepsilon}) > 0$.

Thus by Lemma 4.35 we have

$$Pr_\iota^{\mathcal{M}}\left(\Box(-\mathsf{weight}_{a+\varepsilon} > -1)\right) \geq Pr_\iota^{\mathcal{M}}\left(\Box(-\mathsf{weight}_{a+\varepsilon} > 0)\right) > 0$$

which by Lemma 4.2 yields the desired inequality

$$Pr_\iota^{\mathcal{M}}\left(\Diamond(\mathsf{ratio} > a)\right) = Pr_\iota^{\mathcal{M}}\left(\Diamond(\mathsf{weight}_{a+\varepsilon} > 0)\right) < 1.$$

$\square$

Computing qualitative ratio quantiles in polynomial time

We now have all we need to show that qualitative ratio quantiles are computable in polynomial time. Lemma 5.15 yields that for the long-run modalities $\heartsuit \in \{\Diamond\Box, \Box\Diamond\}$ the qualitative quantiles can be read off the long-run ratios of the BSCCs. As both, the BSCCs of a Markov chain as well as their long-run ratio, can be computed in polynomial time, this directly yields that for $\heartsuit \in \{\Diamond\Box, \Box\Diamond\}$ the qualitative $\heartsuit$-ratio quantiles can be computed in polynomial time.

For the remaining modalities $\Box$ and $\Diamond$, Theorem 5.9, Lemma 5.10, Lemma 5.11 and Lemma 5.13 yield that the following indeed constitutes a polynomial-time computation procedure for qualitative $\heartsuit$-ratio quantiles where $\heartsuit \in \{\Box, \Diamond\}$.

1. Compute the largest denominator N of any element in Q_{all}.
2. Compute an approximation α of the quantile up to precision $1/(2N^2)$.
3. Solve the best-approximation problem for N and α.

We can thus conclude the following theorem.

Theorem 5.21. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ be an energy-utility Markov chain and $\phi = (2^{AP})^\omega$. For $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ the qualitative $\heartsuit$-ratio quantiles are computable in time polynomial in the size of $\mathcal{M}$.*

6 Omega-regular side constraints

In this chapter we will generalize the results of the previous two chapters for omega-regular properties. We show that for an omega-regular side constraint ϕ encoded by a deterministic Rabin automaton $\mathcal{A}$ the qualitative weight and ratio problems are decidable in time polynomial in the size of the Markov chain and $\mathcal{A}$. Further, we prove analogous results for qualitative weight and ratio quantiles. In Section 6.1 we consider the almost-sure case for both decision problems and quantiles, and exploit that for any two events A and B

$$Pr(A \cup B) = 1 \qquad \text{if and only if} \qquad Pr(A) = 1 \text{ and } Pr(B) = 1. \tag{6.1}$$

In contrast, the argument for the positive decision problems and quantiles presented in Section 6.2 relies on the standard reduction of the model-checking problem for ϕ and a Markov chain $\mathcal{M}$ to checking reachability of specific BSCCs for the product $\mathcal{M} \otimes \mathcal{A}$.

6.1 Almost-sure decision problems and quantiles

As mentioned above, the argument for the almost-sure case is based on the observation stated in Equation (6.1).

Lemma 6.1. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ be a weighted Markov chain and ϕ an omega-regular property for $\mathcal{M}$ encoded by a deterministic Rabin automaton $\mathcal{A}$.*

For $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ the almost-sure $\heartsuit$-weight problem is decidable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$ and the almost-sure $\heartsuit$-weight quantile is computable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$.

Proof. Let us first consider the decision problem. Equation (6.1) implies

$$Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) = 1 \quad \text{iff} \quad Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z)) = 1 \text{ and } Pr_\iota^{\mathcal{M}}(\phi) = 1.$$

Using Theorem 4.41, this yields the claim, as it is decidable in polynomial time in the size of $\mathcal{A}$ whether $Pr_\iota^{\mathcal{M}}(\phi) = 1$ (cf. Section 2.2).

Further the above equation implies

$$\mathsf{Qu}_\phi^{=1}[\heartsuit\mathsf{weight}] = \begin{cases} -\infty & \text{if } Pr_\iota^{\mathcal{M}}(\phi) < 1 \\ \mathsf{Qu}^{=1}[\heartsuit\mathsf{weight}] & \text{otherwise} \end{cases}$$

Applying Theorem 5.6 completes the proof. $\square$

Notice that the only weight-specific arguments of Lemma 6.1's proof are the references to Theorem 4.41 and Theorem 5.6. Thus, applying the analogous results for ratios, Corollary 4.42 and Theorem 5.21, we can straight-forwardly extend the argument of the above proof to energy-utility Markov chains.

Lemma 6.2. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ be an energy-utility Markov chain and ϕ an omega-regular property for $\mathcal{M}$ encoded by a deterministic Rabin automaton $\mathcal{A}$.*

For $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ the almost-sure $\heartsuit$-ratio problem is decidable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$ and the almost-sure $\heartsuit$-ratio quantile is computable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$.

6.2 Positive decision problems and quantiles

Let us now consider the positive decision problems and quantiles. As in the previous section both, the weight and the ratio constraints can be treated using the same arguments. As a consequence we will here restrict ourselves to the weighted case and, when necessary, mention the extension to energy-utility Markov chains and ratios.

For the remainder of the section, let us fix a weighted $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ and a DRA $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, Acc)$ accepting the omega-regular property ϕ. In the following we assume $Pr_\iota^{\mathcal{M}}(\phi) > 0$ as otherwise for every integer z and $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$

$$Pr_\iota^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) = 0 \qquad \text{and} \qquad \mathsf{Qu}_\phi^{>0}[\heartsuit(\mathsf{weight} > z)] = -\infty.$$

We show that the results of Theorem 4.41 and Theorem 5.6 can be lifted to arbitrary omega-regular properties based on the following idea. We construct a Markov chain $\mathcal{M}^\phi = (S^\phi, P^\phi, \iota^\phi, \mathsf{weight}^\phi)$ in polynomial time, which satisfies for every integer $z \in \mathbb{Z}$ and $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$

$$Pr_\iota^{\mathcal{M}}(\heartsuit\mathsf{weight} > z \wedge \phi) > 0 \qquad \text{iff} \qquad Pr_{\iota^\phi}^{\mathcal{M}^\phi}(\heartsuit\mathsf{weight} > z) > 0.$$

To do so, let us first extend the product for Markov chains and DRA (cf. Definition 2.14 in Section 2.2) to weighted Markov chains. For $\mathcal{M}$ and $\mathcal{A}$ let the product Markov chain $\mathcal{M} \otimes \mathcal{A} = (S', P', \iota', AP', L', \mathsf{weight}')$ where S', P', ι', AP' and L' are defined as for the standard product and

$$\mathsf{weight}'(\langle s, q\rangle, \langle s', q'\rangle) = \mathsf{weight}(s, s').$$

Let $\mathfrak{C}_{acc}$ be the set of all BSCCs $\mathcal{C}$ of $\mathcal{M} \otimes \mathcal{A}$ which for some $(L, K) \in Acc$ satisfy

$$\mathcal{C} \cap (S \times L) = \varnothing \qquad \text{and} \qquad \mathcal{C} \cap (S \times K) \neq \varnothing.$$

Using standard arguments for omega-regular properties, the definition of weight' implies for every integer $z \in \mathbb{Z}$ and $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$

$$Pr_\iota^{\mathcal{M}}(\heartsuit\mathsf{weight} > z \wedge \phi) > 0 \qquad \text{iff} \qquad Pr_{\iota'}^{\mathcal{M}\otimes\mathcal{A}}(\heartsuit\mathsf{weight} > z \wedge \phi') > 0$$

where ϕ' stands for the set of all paths eventually reaching a BSCC of $\mathfrak{C}_{acc}$.

Remark 6.3. We can directly transfer the argument to energy-utility Markov chains. If we extend the product for both reward functions energy and $\mathsf{utility}$ in the same way as for weight we have

$$Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{ratio} > z) \wedge \phi) > 0 \qquad \text{iff} \qquad Pr_{\iota'}^{\mathcal{M}\otimes\mathcal{A}}(\heartsuit(\mathsf{ratio} > z) \wedge \phi') > 0$$

for every integer $z \in \mathbb{Z}$ and $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$, where ϕ' stands for the set of all paths eventually reaching a BSCC of $\mathfrak{C}_{acc}$.

We now use the product Markov chain $\mathcal{M} \otimes \mathcal{A}$ to construct $\mathcal{M}^{\phi}$. The basic idea of this construction is to "cut off" all BSCCs, which are not contained in $\mathfrak{C}_{acc}$.

However, there are two hickups to consider. First, we need to ensure that we only remove BSCCs and not accidentally create new ones. Here the assumption that $Pr_{\iota}^{\mathcal{M}}(\phi) > 0$ plays a crucial role as it ensures that there exists at least one BSCC $\mathcal{C} \in \mathfrak{C}_{acc}$ which is reachable from ι'. Second, simply removing transitions might violate the requirement that $\mathcal{M}^{\phi}$ constitutes a weighted Markov chain, i.e., that the probabilities for all outgoing transitions of a given state sum up to 1. However, notice that the characterizations of Chapter 4 just consider the existence (or absence) of finite paths reaching a BSCC with a specific expected weight and not their exact probability. Thus, for all transitions not contained in a BSCC, the exact probability distribution is negligible and we can choose any transition preserving distribution.

Definition 6.4. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ be a weighted Markov chain, $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, Acc)$ a DRA encoding an omega-regular property ϕ and their product $\mathcal{M} \otimes \mathcal{A} = (S', P', \iota', AP', L', \mathsf{weight}')$.*

We call the weighted Markov chain $\mathcal{M}^{\phi} = (S^{\phi}, \iota', P^{\phi}, \mathsf{weight}^{\phi})$ a ϕ-restriction of *$\mathcal{M}$ if*

$$\begin{aligned}
S^{\phi} &= \{s \in S' : \ \ \textit{there exists a BSCC in } \mathfrak{C}_{acc} \textit{ reachable from } s\} \\
P^{\phi}(s, s') &= P'(s, s') \textit{ for } s, s' \in \mathfrak{C}_{acc} \\
P^{\phi}(s, s') &> 0 \textit{ iff } P(s, s') > 0 \textit{ for } s \in S^{\phi} \setminus \mathfrak{C}_{acc}, s' \in S^{\phi} \\
\mathsf{weight}^{\phi}(s, s') &= \mathsf{weight}'(s, s') \textit{ for all } s, s' \in S^{\phi}
\end{aligned}$$

Lemma 6.5. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ be a weighted Markov chain, ϕ an omega-regular property and $\mathcal{M}^{\phi} = (S^{\phi}, \iota', P^{\phi}, \mathsf{weight}^{\phi})$ a ϕ-restriction of $\mathcal{M}$. Further let $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$ and $z \in \mathbb{Z}$.*

$$Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) > 0 \qquad \textit{iff} \qquad Pr_{\iota'}^{\mathcal{M}^{\phi}}(\heartsuit(\mathsf{weight} > z)) > 0$$

Proof. Assume that $\mathcal{A}$ is a DRA encoding ϕ such that $\mathcal{M}^{\phi}$ satisfies the requirements for a ϕ-restriction of $\mathcal{M}$ with respect to $\mathcal{A}$ (cf. Definition 6.4) and let $\mathcal{M} \otimes \mathcal{A} = (S', P', \iota', AP', L', \mathsf{weight}')$. It is known that

$$Pr_{\iota}^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z) \wedge \phi) > 0 \qquad \text{iff} \qquad Pr_{\iota'}^{\mathcal{M}\otimes\mathcal{A}}(\heartsuit(\mathsf{weight} > z) \wedge \phi') > 0,$$

where ϕ' contains all paths reaching a BSCC of $\mathcal{M} \otimes \mathcal{A}$ contained $\mathfrak{C}_{acc}$ (cf. Section 2.2). It thus suffices to show

$$Pr_{\iota'}^{\mathcal{M}\otimes\mathcal{A}}(\heartsuit(\mathsf{weight} > z) \wedge \phi') > 0 \qquad \text{iff} \qquad Pr_{\iota^\phi}^{\mathcal{M}^\phi}(\heartsuit(\mathsf{weight} > z)) > 0. \tag{6.2}$$

Notice, every BSCC $\mathcal{C}$ of $\mathcal{M}^\phi$ constitutes a BSCC of $\mathcal{M} \otimes \mathcal{A}$ contained in $\mathfrak{C}_{acc}$ and vice versa. This is a direct consequence of Definition 6.4 and the initial assumption that $Pr_\iota^{\mathcal{M}}(\phi) > 0$. Thus, for almost all infinite paths π of $\mathcal{M} \otimes \mathcal{A}$

$$\pi \in \mathsf{InfPaths}_{\iota'}^{\mathcal{M}^\phi} \text{ iff } \pi \in \mathsf{InfPaths}_{\iota'}^{\mathcal{M}\otimes\mathcal{A}} \cap \phi'.$$

Let us see why. Almost all paths in $\mathsf{InfPaths}_{\iota'}^{\mathcal{M}^\phi}$ eventually reach a BSCC $\mathcal{C}$ of $\mathcal{M}^\phi$. As every path of $\mathcal{M}^\phi$ is a path of $\mathcal{M} \otimes \mathcal{A}$ and $\mathcal{C} \in \mathfrak{C}_{acc}$, almost all paths of $\mathcal{M}^\phi$ are contained in $\mathsf{InfPaths}_{\iota'}^{\mathcal{M}\otimes\mathcal{A}} \cap \phi'$. On the other hand, all paths in $\mathsf{InfPaths}_{\iota'}^{\mathcal{M}\otimes\mathcal{A}} \cap \phi'$ eventually enter a BSCC of $\mathfrak{C}_{acc}$ and are thus contained in $\mathsf{InfPaths}_{\iota'}^{\mathcal{M}^\phi}$.

Additionally by the definition of weight^ϕ and P^ϕ we have for all BSCCs $\mathcal{C}$ of $\mathcal{M}^\phi$

$$\mathbb{E}_{\mathcal{C}}(\mathsf{weight}^\phi) = \mathbb{E}_{\mathcal{C}}(\mathsf{weight}').$$

Using the results of Chapter 4 this yields Equation (6.2) and hence completes the proof, as the characterizations for qualitative weight problems (cf. Lemma 4.32, Lemma 4.33, Lemma 4.34, Lemma 4.35) just consider the existence (or absence) of certain finite paths reaching BSCCs with a specific expected weight and not their probabilities.

□

As $\mathcal{M}^\phi$ is constructible in time linear in the size of $\mathcal{M}\otimes\mathcal{A}$ and the product Markov chain $\mathcal{M} \otimes \mathcal{A}$ is constructible in time polynomial in the size of the Markov chain $\mathcal{M}$ and $\mathcal{A}$, the following theorem is a direct consequence of Theorem 4.41, Theorem 5.6 and Lemma 6.5.

Theorem 6.6. *Let $\mathcal{M}$ be a weighted Markov chain, $\mathcal{A}$ a DRA encoding the linear property ϕ, $z \in \mathbb{Z}$ and $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$. The qualitative $\heartsuit$-weight problems for $\mathcal{M}$, ϕ and z are decidable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$. Further, the qualitative $\heartsuit$-weight quantiles for $\mathcal{M}$, ϕ and z are computable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$.*

As both Definition 6.4 and Lemma 6.5 can be straight-forwardly extended for energy-utility Markov chains, using Corollary 4.42 and Theorem 5.21 we can state the following analogon for Theorem 6.6.

Theorem 6.7. *Let $\mathcal{M}$ be an energy-utility Markov chain, $\mathcal{A}$ a DRA encoding the linear-time property ϕ, $q \in \mathbb{Q}$ and $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$. The qualitative $\heartsuit$-ratio problems for $\mathcal{M}$, ϕ and q are decidable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$. Further, the qualitative $\heartsuit$-ratio quantiles for $\mathcal{M}$, ϕ and q are computable in time polynomial in the size of $\mathcal{M}$ and $\mathcal{A}$.*

7 Extending the results for the strong-release modality

Reconsider the system described in Definition 3.1, which charges and drains a battery. Assume that during the initialization phase (modeled by the states ι, t_1, t_2, and t_3) the system is connected to an external power source. If we want to ensure that the system does not run out of battery, we need to consider the following event:

$$E : \text{the system does not run out of battery after reaching } t_3$$

Notice that the event E cannot be expressed using $\heartsuit(\mathsf{weight} \bowtie z) \wedge \phi$, where $\heartsuit \in \{\Box, \Diamond, \Diamond\Box, \Box\Diamond\}$, $\bowtie \in \{<, \leq, =, \geq, >\}$, $z \in \mathbb{Z}$ and ϕ is an ω-regular property. In fact, E is closely related to the release modality. The weight requirement is released as soon as the system has entered state s_3. However, the standard release modality would allow for paths, which are stuck in the initialization phase.

Inspired by the weak-until modality, in this chapter we introduce the *strong-release* modality and show that we can extend our results to this modality using the same techniques as presented in Chapter 4 and Chapter 5. In Section 7.1 we consider both qualitative decision problems and quantiles related to strong release and weighted Markov chains, before we turn to energy-utility Markov chains in Section 7.2.

7.1 Strong release and weighted Markov chains

In the section we will consider qualitative decision problems and quantiles related to the following events. Given a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$, a set of states $\mathsf{goal} \subseteq S$, an integer $z \in \mathbb{Z}$ and $\bowtie \in \{<, \leq, =, \geq, >\}$ the events $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{weight} \bowtie z))$ and $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{weight} \bowtie z))$ are defined by

$$\pi \models (\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{weight} \bowtie z)) \quad \text{iff} \quad \begin{array}{l} \text{there exists } k \in \mathbb{N}_{>0} \text{ s.t. } \pi[k] \in \mathsf{goal}, \\ \pi[n_1] \notin \mathsf{goal} \text{ for all } n_1 \in \mathbb{N}_{<k} \text{ and} \\ \mathsf{weight}(\pi[\ldots n_2]) \bowtie z \text{ for all } n_2 \in \mathbb{N}_{>k}, \end{array}$$

$$\pi \models (\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{weight} \bowtie z)) \quad \text{iff} \quad \begin{array}{l} \text{there exists } k_1, k_2 \in \mathbb{N}_{>0},\ k_1 < k_2 \text{ s.t.} \\ \pi[k_1] \in \mathsf{goal} \text{ and } \mathsf{weight}(\pi[\ldots k_2]) \bowtie z. \end{array}$$

We will show that given a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$, a set of states $\mathsf{goal} \subseteq S$, a modality $\heartsuit \in \{\Box, \Diamond\}$, and an integer $z \in \mathbb{Z}$ the following decision problems and quantiles are solvable in polynomial time.

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) > 0$$
$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) = 1$$

$$\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{weight})] = \sup\{z' \in \mathbb{Z} : \; Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z')) > 0\}$$
$$\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{weight})] = \sup\{z' \in \mathbb{Z} : \; Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z')) = 1\}$$

To this end, for the remainder of the section let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$ be a Markov chain, $\mathsf{goal} \subseteq S$ a set of states and $z \in \mathbb{Z}$.

Remark 7.1. We will not consider the analogous events for the modalities $\Diamond\Box$ and $\Box\Diamond$ as they are equivalent to $\Diamond\Box(\mathsf{weight} \bowtie z) \wedge \Diamond\mathsf{goal}$ and $\Box\Diamond(\mathsf{weight} \bowtie z) \wedge \Diamond\mathsf{goal}$. Hence, the results of Chapter 6 yield that for these events both, the decision problems and the quantiles are solvable in polynomial time.

In the following, we assume that ι is not contained in goal as for $\iota \in \mathsf{goal}$ the event $(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{weight} > z)$ is equivalent to $\heartsuit(\mathsf{weight} > z)$ and thus trivially reduces to Theorem 6.6.

The proof for the above decision problems is based on the following idea. Every path π satisfying $(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z))$ can be divided into a finite path $\hat{\pi}$ "activating" the release, i.e, $\mathsf{last}(\hat{\pi}) \in \mathsf{goal}$ and $\hat{\pi}[i] \notin \mathsf{goal}$ for $0 \le i < |\hat{\pi}|$, and a suffix π' such that $\pi = \hat{\pi} \diamond \pi'$. Using standard graph algorithms we can compute both the minimal and maximal weight accumulated on such $\hat{\pi}$. Let $dist_{\neg\mathsf{goal},max}(s_1, s_2)$ be the maximal distance from s_1 to s_2 with respect to the graph induced by $\mathcal{M}$ restricted to the states $S \setminus \mathsf{goal}$. Analogously $dist_{\neg\mathsf{goal},min}(s_1, s_2)$ denotes the minimal distance from s_1 to s_2 with respect to the same graph. For every $s \in \mathsf{goal}$, we define

$$w_{max}(s) := \max_{s'}(dist_{\neg\mathsf{goal},max}(\iota, s') + \mathsf{weight}(s', s)) \quad \text{and}$$
$$w_{min}(s) := \min_{s'}(dist_{\neg\mathsf{goal},min}(\iota, s') + \mathsf{weight}(s', s)),$$

where s' ranges over all predecessors of s not contained in goal, i.e., $s' \notin \mathsf{goal}$ and $P(s', s) > 0$. Using the notion of w_{max} and w_{min} we can now state the characterizations for the decision problems $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) > 0$ (cf. Lemma 7.2) and $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) = 1$ (Lemma 7.3).

Lemma 7.2.

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) > 0 \qquad \textit{if and only if}$$
$$\exists s \in \mathsf{goal} \colon Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{max}(s))) > 0$$

Proof. The implication ($\Leftarrow$) trivially holds. Assume that $\hat{\pi}$ is a finite path from ι to s such that $\mathsf{weight}(\hat{\pi}) = w_{max}$ and $Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{max}(s))) > 0$. Then,

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) > P(\hat{\pi}) \cdot Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{max}(s))) > 0.$$

In order to show implication ($\Rightarrow$), let us first consider modality $\Diamond$. Assume that $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{weight} > z)) > 0$. Thus, there exists a path $\pi \in \mathsf{InfPaths}_\iota^{\mathcal{M}}$ satisfying $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{weight} > z))) > 0$. Let k be the smallest natural number such that $\pi[k] \in \mathsf{goal}$. Hence, there exists an $m > k$ such that

$$\mathsf{weight}(\pi[\ldots m]) = \mathsf{weight}(\pi[\ldots k]) + \mathsf{weight}(\pi[k \ldots m]) > z,$$

where $\pi[k \ldots m]$ stands for $\pi[k \ldots][\ldots (m-k)]$. Let $s = \pi[k]$. By definition of w_{max} we have $\mathsf{weight}(\pi[\ldots k]) \leq w_{max}(s)$, which implies $\pi[k \ldots m] > z - w_{max}(s)$ and hence $\pi[k \ldots] \models \Diamond(\mathsf{weight} > z - w_{max}(s))$. This yields the claim for $\Diamond$ as

$$Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{max}(s))) \geq P(\pi[k \ldots m]) > 0.$$

Let us now turn to modality $\Box$. We assume towards a contradiction that there exists no $s \in \mathsf{goal}$ with $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z - w_{max}(s))) > 0$ but the probability $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{weight} > z))$ is positive. By Corollary 4.11 and Corollary 4.16 there exists a path $\pi \in \mathsf{InfPaths}_\iota^{\mathcal{M}}$ satisfying $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{weight} > z))$ eventually reaching a BSCC $\mathcal{C}$ with either $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) > 0$ or $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}) = 0$ and no negative cycles. Let k be the first ocurrence of a goal-state in π and $s = \pi[k]$, i.e., $\pi[k] = s \in \mathsf{goal}$ and $\pi[n] \notin \mathsf{goal}$ for all $n < k$. Thus for all $m > k$,

$$\mathsf{weight}(\pi[\ldots m]) = \mathsf{weight}(\pi[\ldots k]) + \mathsf{weight}(\pi[k \ldots m]) > z.$$

As $\mathsf{weight}(\pi[\ldots k]) \leq w_{max}(s)$ this implies $\pi[k \ldots] \models \Box\mathsf{weight} > z - w_{max}(s)$. Applying Lemma 4.35 yields $Pr_s^{\mathcal{M}}(\Box(\mathsf{weight} > z - w_{max}(s))) > 0$ and thus the aimed contradiction. $\Box$

Lemma 7.3.

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) = 1 \quad \textit{if and only if}$$
$$Pr_\iota^{\mathcal{M}}(\Diamond\mathsf{goal}) = 1 \textit{ and } \forall s \in \mathsf{goal}\colon Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{min}(s))) = 1$$

Proof. Notice, by the definition of the $\mathcal{R}^\bullet$ operator, $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) = 1$ implies $Pr_\iota^{\mathcal{M}}(\Diamond\mathsf{goal}) = 1$.

($\Rightarrow$): Towards a contradiction assume $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z)) = 1$ and there exists a state $s \in \mathsf{goal}$ such that $Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{min}(s))) < 1$. Thus, $Pr_s^{\mathcal{M}}(\neg(\heartsuit(\mathsf{weight} > z - w_{min}(s)))) > 0$. This yields a contradiction as for any finite path $\hat{\pi} \in \mathsf{FinPaths}_{\iota \ldots s}^{\mathcal{M}}$ with $\mathsf{weight}(\hat{\pi}) = w_{min}(s)$

$$Pr_\iota^{\mathcal{M}}(\neg(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{weight} > z))) > P(\hat{\pi}) \cdot Pr_s^{\mathcal{M}}(\neg(\heartsuit(\mathsf{weight} > z - w_{min}(s)))) > 0.$$

($\Leftarrow$): Assume that $Pr_\iota^{\mathcal{M}}(\Diamond\mathsf{goal}) = 1$ and for every state $s \in \mathsf{goal}$ we have $Pr_s^{\mathcal{M}}(\heartsuit(\mathsf{weight} > z - w_{min}(s))) = 1$. We argue for $Pr_\iota^{\mathcal{M}}$-almost-all infinite paths π. As $Pr_\iota^{\mathcal{M}}(\Diamond\mathsf{goal}) = 1$, $\pi = \hat{\pi} \diamond \pi'$ for some $\pi' \in \mathsf{InfPaths}^{\mathcal{M}}$ and $\hat{\pi} \in \mathsf{FinPaths}_\iota^{\mathcal{M}}$ such that

$\mathsf{last}(\hat{\pi}) \in \mathsf{goal}$ and for every real prefix ρ of $\hat{\pi}$, $\mathsf{last}(\rho) \notin \mathsf{goal}$. Let $s = \mathsf{last}(\hat{\pi})$. By definition of w_{min} we have $\mathsf{weight}(\hat{\pi}) \geq w_{min}(s)$. Further $\pi' \models \heartsuit(\mathsf{weight} > z - w_{min}(s))$. This completes the argument as, if $\heartsuit = \Box$ for every nonempty prefix ρ of π'

$$\mathsf{weight}(\hat{\pi} \diamond \rho) = \mathsf{weight}(\hat{\pi}) + \mathsf{weight}(\rho) > z$$

and in case $\heartsuit = \lozenge$ there exists a prefix ρ satisfying the above inequality. $\Box$

Applying Theorem 6.6, Lemma 7.2 and Lemma 7.3 directly yield that the considered decision problems are decidable in polynomial time.

Lemma 7.4. *Given a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$, a set of states* goal *and an integer z. For $\heartsuit \in \{\Box, \lozenge\}$ it is decidable in time polynomial in the size of $\mathcal{M}$ whether the following statements hold:*

$$Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit(\mathsf{weight} > z)) > 0 \quad \textit{and} \quad Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit(\mathsf{weight} > z)) = 1.$$

In fact, Lemma 7.2 and Lemma 7.3 provide a polynomial computation scheme for the qualitative quantiles $\mathsf{Qu}^{>0}[(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit\mathsf{weight})]$ and $\mathsf{Qu}^{=1}[(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit\mathsf{weight})]$. For a given $s \in S$ let $\mathsf{Qu}_s^{>0}[\heartsuit\mathsf{weight}]$ be the positive $\heartsuit$-weight quantile with respect to $(S, P, s, \mathsf{weight})$. By the definition of quantiles, for any integer $w \in \mathbb{Z}$, $\mathsf{Qu}_s^{>0}[\heartsuit\mathsf{weight}] + w$ is the largest integer w' such that $Pr_s^{\mathcal{M}}(\heartsuit\mathsf{weight} > w' - w) > 0$. Hence, we can conclude the following two lemmata.

Lemma 7.5. *Let $q_s = \mathsf{Qu}_s^{>0}[\heartsuit\mathsf{weight}]$ the positive $\heartsuit$-weight quantile with respect to $(S, P, s, \mathsf{weight})$.*

$$\mathsf{Qu}^{>0}[(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit\mathsf{weight})] = \max_{s \in \mathsf{goal}} (q_s + w_{max}(s))$$

Lemma 7.6. *Let $q_s = \mathsf{Qu}_s^{=1}[\heartsuit\mathsf{weight}]$ the almost-sure $\heartsuit$-weight quantile with respect to $(S, P, s, \mathsf{weight})$.*

$$\mathsf{Qu}^{=1}[(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit\mathsf{weight})] = \begin{cases} -\infty & \textit{if } Pr_{\iota}^{\mathcal{M}}(\lozenge\mathsf{goal}) < 1 \\ \min_{s \in \mathsf{goal}}(q_s + w_{min}(s)) & \textit{otherwise} \end{cases}$$

Again, applying Theorem 6.6 yield the desired result.

Lemma 7.7. *Given a weighted Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{weight})$, a set of states* goal *and an integer z. For $\heartsuit \in \{\Box, \lozenge\}$ the following quantiles are computable in polynomial time:*

$$\mathsf{Qu}^{>0}[(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit\mathsf{weight})] \quad \textit{and} \quad \mathsf{Qu}^{=1}[(\mathsf{goal}\, \mathcal{R}^{\bullet}\, \heartsuit\mathsf{weight})].$$

7.2 Strong release and energy-utility Markov chains

In this section, we show that the result of the previous section can be lifted to energy-utility Markov chains.

To this end, let us first fix the notation. Given an energy-utility Markov chain $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$, a set of states $\mathsf{goal} \subseteq S$, a threshold $q \in \mathbb{Q}$ and $\bowtie \in \{<, \leq, =, \geq, >\}$, the events $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} \bowtie q))$ and $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{ratio} \bowtie q))$ are defined analogous to the weighted case:

$$
\begin{array}{lll}
\pi \models (\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} \bowtie q)) & \text{iff} & \text{there exists } k \in \mathbb{N}_{>0} \text{ s.t. } \pi[k] \in \mathsf{goal}, \\
& & \pi[n_1] \notin \mathsf{goal} \text{ for all } n_1 \in \mathbb{N}_{<k} \text{ and} \\
& & \mathsf{ratio}(\pi[\ldots n_2]) \bowtie q \text{ for all } n_2 \in \mathbb{N}_{>k}, \\
\pi \models (\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{ratio} \bowtie q)) & \text{iff} & \text{there exist } k_1, k_2 \in \mathbb{N}_{>0}, k_1 < k_2 \text{ s.t.} \\
& & \pi[k_1] \in \mathsf{goal} \text{ and } \mathsf{ratio}(\pi[\ldots k_2]) \bowtie q.
\end{array}
$$

As in Chapter 4 (cf. Lemma 4.1) we can reduce the qualitative decision problems $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{ratio} > q)) > 0$ and $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{ratio} > q)) = 1$ to the associated decision problems for weighted Markov chains.

Lemma 7.8. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ be an energy-utility Markov chain, $\mathsf{goal} \subseteq S$ a set of states and $q_1, q_2 \in \mathbb{N}$. Further, as before let $\mathsf{weight}_{q_1/q_2} = q_2 \cdot \mathsf{utility} - q_1 \cdot \mathsf{energy}$. For $\heartsuit \in \{\Box, \Diamond\}$ and $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$*

$$\pi \models (\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{ratio} > (q_1/q_2)) \qquad \textit{iff} \qquad \pi \models (\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{weight}_{q_1/q_2} > 0).$$

Proof. Directly follows from the following fact. For every finite path $\hat{\pi}$ we have $\mathsf{ratio}(\hat{\pi}) > (q_1/q_2)$ if and only if $\mathsf{weight}_{(q_1/q_2)}(\hat{\pi}) > 0$. $\Box$

Corollary 7.9. *Let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ be an energy-utility Markov chain, $\mathsf{goal} \subseteq S$ a set of states and $q \in \mathbb{Q}_{\geq 0}$. For $\heartsuit \in \{\Box, \Diamond\}$ it is decidable in time polynomial in the size of $\mathcal{M}$ whether the following statements hold:*

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{ratio} > q)) > 0 \qquad \textit{and} \qquad Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{ratio} > q)) = 1.$$

Let us now consider quantiles with respect to the $\mathcal{R}^\bullet$ modality and energy-utility Markov chains. To this end, let $\mathcal{M} = (S, P, \iota, AP, L, \mathsf{utility}, \mathsf{energy})$ be an energy-utility Markov chain. Further, we fix a set of states $\mathsf{goal} \subseteq S$ and a threshold $q \in \mathbb{Q}$. Let us define the following qualitative quantiles for $\heartsuit \in \{\Box, \Diamond\}$.

$$
\begin{aligned}
\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{ratio}))] &= \sup\{q \in \mathbb{Q} : \ Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{ratio} > q)) > 0\} \\
\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{ratio}))] &= \sup\{q \in \mathbb{Q} : \ Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit(\mathsf{ratio} > q)) = 1\}
\end{aligned}
$$

As already discussed in Section 5.2, Lemma 7.8 does not allow to reduce qualitative ratio quantiles to qualitative weight quantiles. To show that the quantiles $\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{ratio}))]$ and $\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\heartsuit\mathsf{ratio}))]$ are computable in polynomial

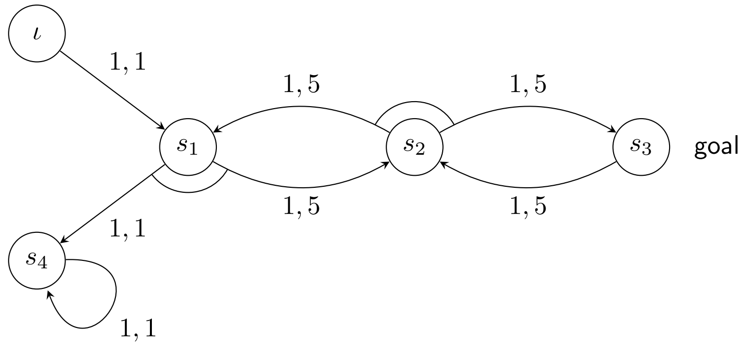

Figure 7.1: Energy-utility Markov chain (transitions are labeled with tuples $(\mathsf{utility}, \mathsf{energy})$ and choices are assumed to be uniformly distributed)

time we will slightly adapt the argument for "standard" qualitative ratio quantiles presented in Section 5.2.1. We there have shown that it suffices to prove the following two claims in order to provide a polynomial time computation scheme.

(I) Given $N \in \mathbb{N}$ we can approximate the quantiles $\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio}))]$ and $\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio}))]$ up to precision $1/(2N^2)$ in polynomial time.

(II) We can compute an upper bound for the denominator of the quantile value in polynomial time.

To show the first claim, we can straight-forwardly adapt the proof of Lemma 5.11.

Lemma 7.10. *Given $N \in \mathbb{N}$ and $\heartsuit \in \{\Box, \Diamond\}$, the quantiles $\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio}))]$ and $\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio}))]$ can be approximated up to precision $1/(2N^2)$ in time polynomial in the encoding length of N and the size of $\mathcal{M}$.*

To show the second claim, we can adapt the proof of Lemma 5.12 to show the following lemma.

Lemma 7.11. *One can compute a natural number N in polynomial time such that for every $\heartsuit \in \{\Box, \Diamond\}$ and $* \in \{> 0, = 1\}$*

$$\mathsf{Qu}^{*}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio}))] \in \{a/b : \ a \in \mathbb{N} \ \textit{and} \ \in [1, N]\}.$$

Remember, in order to prove Lemma 5.12 we showed that $Q_{all} = Q_{bscc} \cup Q_{cycle} \cup Q_{path}$ serves as a candidate set for the qualitative ratio quantiles (cf. Lemma 5.13). The above lemma can be proven using the same arguments, however, we need to consider a slightly different set. To get an intuition, let us first consider why Q_{all} does not work.

Example 7.12. Consider the energy-utility Markov chain $\mathcal{M}$ depicted in Figure 7.1 and let $\mathsf{goal} = \{s_3\}$. We have $\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box\mathsf{ratio})] = 5/21 \notin Q_{all}$ as there exists no $\pi \in \mathsf{InfPaths}_{\iota}^{\mathcal{M}}$ with $\pi \models (\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{ratio} > 5/21))$, but for every $\varepsilon \in \mathbb{Q}_{>0}$

$$\iota s_1 s_2 s_3 s_2 s_1 s_4^{\omega} \models (\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{ratio} > (5/21) - \varepsilon)).$$

As already mentioned above, every infinite path $\pi \in \mathsf{InfPaths}^{\mathcal{M}}$ such that $\pi \models (\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{ratio} > q))$ can be decomposed into a finite path reaching a state in goal, a finite path starting in a goal-state and reaching a BSCC and a suffix, e.g., if we reconsider the path $\iota s_1 s_2 s_3 s_2 s_1 s_4^{\omega}$ of Example 7.12

$$\iota s_1 s_2 s_3 s_2 s_1 s_4^{\omega} = \iota s_1 s_2 s_3 \diamond s_3 s_2 s_1 s_4 \diamond s_4^{\omega}.$$

Using this decomposition, we can slightly amend the argument for Lemma 5.18 and Lemma 5.20 in order to show that the quantiles related to strong-release are contained in the $Q_{release} = Q_{bscc} \cup Q_{cycle} \cup Q_{2|S|path}$, where $Q_{2|S|path}$ contains the ratios of all finite paths with length at most $2 \cdot |S|$, i.e.,

$$Q_{2|S|path} = \{\mathsf{ratio}(\hat{\pi}) : \ \hat{\pi} \in \mathsf{FinPaths}_{\iota}^{\mathcal{M}} \text{ such that } |\hat{\pi}| < 2 \cdot |S|\}.$$

Before we, for the sake of completeness, rephrase the arguments presented in the proof of Lemma 5.18 and Lemma 5.20, let us discuss an upper bound for the denominators in $Q_{release}$. Such an upper bound can be computed in polynomial time as $2 \cdot |S| \max_{(s,s') \in S^2} \mathsf{energy}(s, s')$ is an upper bound for both Q_{cycle} and $Q_{2|S|path}$. Additionally, Q_{bscc} is computable in polynomial time.

Lemma 7.13. *Let* $* \in \{> 0, = 1\}$ *and* $\heartsuit \in \{\Box, \Diamond\}$.

$$\mathsf{Qu}^*[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio})] \in Q_{release}$$

We dedicate the remainder of the section to the proof of the above lemma. To this end, we will consider auxiliary Markov chains, which differ only in their initial state and weight functions, i.e., $(S, P, s, \mathsf{weight}_{q_1/q_2})$ where $s \in S$, $q_1, q_2 \in \mathbb{Q}$ and as before

$$\mathsf{weight}_{q_1/q_2} = q_2 \cdot \mathsf{utility} - q_1 \cdot \mathsf{energy}.$$

Several of our Markov chain related notations such as *cdist* or w_{min} are defined with respect to a given Markov chain. Whenever we refer to such a notion with respect to an auxiliary Markov chain, we will state the considered initial state and weight function in brackets, i.e., $w_{min}[s, \mathsf{weight}_q](s')$ denotes $w_{min}(s')$ with respect to $(S, P, s, \mathsf{weight}_q)$.

In the following we will show $\mathsf{Qu}^*[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio})] \in Q_{release}$ separately for every combination $* \in \{> 0, = 1\}$ and $\heartsuit \in \{\Box, \Diamond\}$. In fact, we will rephrase and slightly amend the arguments presented in the proofs of Lemma 5.18 and Lemma 5.20. The proofs all refer to the following simple observation.

Lemma 7.14. *Let* $* \in \{> 0, = 1\}$ *and* $\heartsuit \in \{\Box, \Diamond\}$.

$$\min Q_{release} \leq \mathsf{Qu}^*[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\heartsuit\mathsf{ratio})] \leq \max Q_{release}$$

Proof. It suffices to show that for every finite path $\hat{\pi}$,

$$\min Q_{release} \leq \mathsf{ratio}(\hat{\pi}) \leq \max Q_{release}.$$

Notice that we have already shown this claim in the proof of Lemma 5.16 (cf. part (b)). $\Box$

Lemma 7.15.

$$\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box\mathsf{ratio})] \in Q_{release}$$

Proof. The argument is analogous as for Lemma 5.18. Applying Lemma 7.14 it suffices to show the following claim:

Let $a, b \in Q_{release}$ be such that $a < b$ and there is no $m \in Q_{release}$ satisfying $a < m < b$. Let $c \in \mathbb{Q}$ such that $a < c < b$. If $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{ratio} > c)) > 0$, then for every $\varepsilon \in \mathbb{Q}_{>0}$, $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{ratio} > b - \varepsilon)) > 0$.

Applying Lemma 7.8 we can conclude that $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{ratio} > c)) > 0$ is equivalent to $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_c > 0)) > 0$. The basic idea of the proof is to use a (specific) path satisfying $(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_c > 0))$ to show that both conditions of the characterization for $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_{b-\varepsilon} > 0)) > 0$ (cf. Lemma 7.2) are satisfied. Applying Lemma 7.8 again yields the claim.

By Corollary 4.11 and Corollary 4.16, $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_c > 0)) > 0$ implies that there exists a path π satisfying $(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_c > 0))$ and reaching a BSCC $\mathcal{C}$ with $\mathbb{E}_{\mathcal{C}}(\mathsf{ratio}) \geq c$. As $\mathbb{E}_{\mathcal{C}}(\mathsf{ratio}) \in Q_{bscc} \subseteq Q_{release}$ this yields $\mathbb{E}_{\mathcal{C}}(\mathsf{ratio}) \geq b$. Let $k \in \mathbb{N}$ be the smallest position such that $\pi[k] \in \mathsf{goal}$. We can safely assume that $\pi[k\ldots]$ contains at least one cycle ϑ with $\mathsf{weight}_c(\vartheta) \geq 0$ and $\pi[\ldots k]$ contains only weight_c-positive simple cycles. We now consider two cases.

$\pi[\ldots k]$ *contains a cycle:* Hence, there exists $\hat{\pi}_1, \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}$ and a weight_c-positive simple cycle ϑ such that $\pi[\ldots k] = \hat{\pi}_1 \diamond \vartheta \diamond \hat{\pi}_2$. As $\mathsf{weight}_c(\vartheta) > 0$ is equivalent to $\mathsf{ratio}(\vartheta) > c$ and further $\mathsf{ratio}(\vartheta) \in Q_{cycle} \subseteq Q_{release}$, ϑ is a $\mathsf{weight}_{b-\varepsilon}$-positive cycle. Notice that this implies $w_{max}[\iota, \mathsf{weight}_{b-\varepsilon}](\pi[k]) = \infty$. Further $\mathbb{E}_{\mathcal{C}}(\mathsf{ratio}) \geq b$ implies $\mathbb{E}_{\mathcal{C}}(\mathsf{weight}_{b-\varepsilon}) > 0$ and thus by Lemma 4.35 there exists an $n \in \mathbb{N}$ such that $Pr_{\pi[k]}^{\mathcal{M}}(\Box\mathsf{weight}_{b-\varepsilon} > -n) > 0$. Hence, using Lemma 7.2 we can conclude $Pr_{\iota}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_{b-\varepsilon} > 0)) > 0$.

$\pi[\ldots k]$ *is a simple path:* Similar as in the proof of Lemma 5.18 we will use $\pi[k\ldots]$ to construct a witness for

$$Pr_{\pi[k]}^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box(\mathsf{weight}_{b-\varepsilon} > -w_{max}[\iota, \mathsf{weight}_{b-\varepsilon}](\pi[k]))) > 0. \tag{7.1}$$

Let $m \in \mathbb{N}$ be the smallest position $m > k$ such that $\pi[m] \in \mathcal{C}$ and $\pi[\ldots m]$ contains a weight_c-positive cycle. Thus, there exist $\hat{\pi}_1, \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}$ and a simple weight_c-positive cycle ϑ such that $\pi[k\ldots m] = \hat{\pi}_1 \diamond \vartheta \diamond \hat{\pi}_2$. Notice that ϑ is a $\mathsf{weight}_{b-\varepsilon}$-positive cycle as $\mathsf{weight}_c(\vartheta) > 0$ is equivalent to $\mathsf{ratio}(\vartheta) > c$ and $\mathsf{ratio}(\vartheta) \in Q_{cycle} \subseteq Q_{release}$. Further we can safely assume that every nonempty prefix $\rho \neq \hat{\pi}_1 \diamond \vartheta$ of $\hat{\pi}_1 \diamond \vartheta$ is a simple path.

Thus, for every nonempty prefix $\rho \neq \hat{\pi}_1 \diamond \vartheta$ of $\hat{\pi}_1 \diamond \vartheta$ we have $|\pi[\ldots k] \diamond \rho| < 2 \cdot |S|$ and hence $\mathsf{ratio}(\pi[\ldots k] \diamond \rho) \in Q_{release}$. Further, $\mathsf{ratio}(\pi[\ldots k] \diamond \rho) > c$ as $\pi \models (\mathsf{goal}\,\mathcal{R}^{\bullet}\,\Box\mathsf{weight}_c > 0)$. Using the same argument as for ϑ, we can conclude $\mathsf{ratio}(\pi[\ldots k] \diamond \rho) > b - \varepsilon$ and $\mathsf{weight}_{b-\varepsilon}(\pi[\ldots k] \diamond \rho) > 0$. Hence,

$$\mathsf{weight}_{b-\varepsilon}(\rho) = \mathsf{weight}_{b-\varepsilon}(\pi[\ldots k] \diamond \rho) - \mathsf{weight}(\pi[\ldots k]) > -w_{max}(\pi[k]).$$

As ϑ is a $\mathsf{weight}_{b-\varepsilon}$-positive cycle we have $cdist[\pi[k], \mathsf{weight}_{b-\varepsilon}](\pi[m], -w_{max}) = \infty$. Thus, Lemma 4.35 yields Equation (7.1) and Lemma 7.2 completes the argument.

□

Lemma 7.16. *Let $Pr_\iota^{\mathcal{M}}(\Diamond \mathsf{goal}) = 1$.*

$$\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box \mathsf{ratio})] \in Q_{release}$$

Proof. We show the following claim:

Let $a, b \in Q_{release}$ such that $a < b$ and there is no element $m \in Q_{release}$ satisfying $a < m < b$. Let $c \in \mathbb{Q}$ such that $a < c < b$. If $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} > c)) < 1$, then for every $\varepsilon \in \mathbb{Q}_{>0}$ $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} > a + \varepsilon)) < 1$.

Let $\pi \in \mathsf{InfPaths}_\iota^{\mathcal{M}}$ be a path satisfying $\neg(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} > c))$ and $\Diamond\mathsf{goal}$. Further let $k \in \mathbb{N}$ be the smallest position such that $\pi[k] \in \mathsf{goal}$ and $m > k$ the smallest position such that $\mathsf{ratio}(\pi[\ldots m]) \leq c$. We can safely assume that both $\pi[\ldots k]$ and $\pi[k \ldots m]$ contain only simple cycles ϑ with $\mathsf{ratio}(\vartheta) \leq c$. We now consider two cases.

$\pi[\ldots k]$ or $\pi[k \ldots m]$ contains a cycle: Hence, there exist $\hat{\pi}_1, \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}$ and a simple cycle ϑ such that $\mathsf{ratio}(\vartheta) \leq c$ and $\pi[\ldots m] = \hat{\pi}_1 \diamond \vartheta \diamond \hat{\pi}_2$. Further as $\mathsf{ratio}(\vartheta) \in Q_{cycle} \subseteq Q_{release}$, we have $\mathsf{ratio}(\vartheta) \leq a$. Thus for every $\varepsilon \in \mathbb{Q}_{>0}$ there exists an $n \in \mathbb{N}$ such that $\mathsf{ratio}(\hat{\pi}_1 \diamond (\vartheta)^n \diamond \hat{\pi}_2) < a + \varepsilon$. This completes the argument, as

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} > a + \varepsilon)) \leq 1 - Pr(\hat{\pi}_1 \diamond (\vartheta)^n \diamond \hat{\pi}_2) < 1.$$

both $\pi[\ldots k]$ and $\pi[k \ldots m]$ are simple paths: Thus $|\pi[\ldots m]| < 2 \cdot |S|$ and therefore $\mathsf{ratio}(\pi[\ldots m]) \in Q_{release}$. This yields the claim as it implies $\mathsf{ratio}(\pi[\ldots m]) \leq a$ and hence

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Box(\mathsf{ratio} > a + \varepsilon)) \leq 1 - Pr(\pi[\ldots m]) < 1.$$

□

Lemma 7.17.

$$\mathsf{Qu}^{>0}[(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond \mathsf{ratio})] \in Q_{release}$$

Proof. We show the following claim:

Let $a, b \in Q_{release}$ such that $a < b$ and there is no element $m \in Q_{release}$ satisfying $a < m < b$. Let $c \in \mathbb{Q}$ such that $a < c < b$. If $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{ratio} > c)) > 0$, then for every $\varepsilon \in \mathbb{Q}_{>0}$ $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{ratio} > b - \varepsilon)) > 0$.

Let $\pi \in \mathsf{InfPaths}_\iota^{\mathcal{M}}$ be a path satisfying $(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{ratio} > c))$. Further let $k \in \mathbb{N}$ be the smallest position such that $\pi[k] \in \mathsf{goal}$ and $m \in \mathbb{N}$ the smallest position $m > k$ such that $\mathsf{ratio}(\pi[\ldots m]) > c$. We can safely assume that both, $\pi[\ldots k]$ and $\pi[k \ldots m]$ contain only simple cycles with $\mathsf{ratio}(\vartheta) > c$. We now consider two cases.

$\pi[\ldots k]$ or $\pi[k \ldots m]$ contain a cycle Hence, there exist $\hat{\pi}_1, \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}$ and a simple cycle ϑ with $\mathsf{ratio}(\vartheta) > c$ and $\pi[\ldots m] = \hat{\pi}_1 \diamond \vartheta \diamond \hat{\pi}_2$. Further as $\mathsf{ratio}(\vartheta) \in Q_{cycle} \subseteq Q_{release}$, we have $\mathsf{ratio}(\vartheta) \geq b$. Thus for every $\varepsilon \in \mathbb{Q}_{>0}$ there exists an $n \in \mathbb{N}$ such that $\mathsf{ratio}(\hat{\pi}_1 \diamond (\vartheta)^n \diamond \hat{\pi}_2) > b - \varepsilon$. This completes the argument, as

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet\,\Diamond(\mathsf{ratio} > b - \varepsilon)) \geq Pr(\hat{\pi}_1 \diamond (\vartheta)^n \diamond \hat{\pi}_2) > 0.$$

both $\pi[\ldots k]$ *and* $\pi[k \ldots m]$ *are simple paths:* Thus $|\pi[\ldots m]| < 2 \cdot |S|$ and hence $\mathsf{ratio}(\pi[\ldots m]) \in Q_{release}$. This completes the argument as it implies $\mathsf{ratio}(\pi[\ldots m]) \geq b$ and therefore

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{ratio} > b - \varepsilon)) \geq P(\pi[\ldots m]) > 0.$$

□

Lemma 7.18. *Let* $Pr_\iota^{\mathcal{M}}(\lozenge \mathsf{goal}) = 1$.

$$\mathsf{Qu}^{=1}[(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge \mathsf{ratio})] \in Q_{release}$$

Proof. We show the following claim using a similar argument as presented in the proof of Lemma 5.20.

Let $a, b \in Q_{release}$ such that $a < b$ and there is no element $m \in Q_{release}$ satisfying $a < m < b$. Let $c \in \mathbb{Q}$ such that $a < c < b$. If $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{ratio} > c)) < 1$, then for every $\varepsilon \in \mathbb{Q}_{>0}$ $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{ratio} > a + \varepsilon)) < 1$.

By Lemma 7.8 and Lemma 7.3 there exists a state $s \in \mathsf{goal}$ with

$$Pr_s^{\mathcal{M}}\left(\lozenge\left(\mathsf{weight}_c > -w_{min}[\iota, \mathsf{weight}_c](s)\right)\right) < 1.$$

Let us consider two cases.

Case 1: $w_{min}[\iota, \mathsf{weight}_c](s) = -\infty$. Thus there exist $\hat{\pi}_1, \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}$ and a weight_c-negative simple cycle ϑ such that $\hat{\pi} = \hat{\pi}_1 \diamond \vartheta \diamond \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}$. Further $\mathsf{weight}_c(\vartheta) < 0$ is equivalent to $\mathsf{ratio}(\vartheta) < c$ and as $\mathsf{ratio}(\vartheta) \in Q_{cycle}$, we have $\mathsf{ratio}(\vartheta) \leq a$ and hence $\mathsf{weight}_{a+\varepsilon}(\vartheta) < 0$. Thus $w_{min}[\iota, \mathsf{weight}_{a+\varepsilon}](s) = -\infty$ and Lemma 7.3 yields

$$Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{weight}_{a+\varepsilon} > 0)) < 1.$$

By Lemma 7.8 we have $Pr_\iota^{\mathcal{M}}(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{ratio} > a + \varepsilon)) < 1$.

Case 2: $w_{min}[\iota, \mathsf{weight}_c](s) \in \mathbb{Z}$. As $w_{min}[\iota, \mathsf{weight}_c](s) \in \mathbb{Z}$ there exists a finite path $\hat{\pi} \in \mathsf{FinPaths}^{\mathcal{M}}_{\iota \ldots s}$ such that $\mathsf{weight}_c(\hat{\pi}) = w_{min}[\iota, \mathsf{weight}_c](s)$ and we can safely assume that all simple cycle of $\hat{\pi}$ are weight_c-negative. Let $\pi \in \mathsf{InfPaths}^{\mathcal{M}}_s$ be an infinite path such that $\hat{\pi} \diamond \pi$ satisfies $\neg(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{ratio} > c))$. In the following we will use π and Lemma 4.35 to show

$$Pr_s^{\mathcal{M}}(\Box(-\mathsf{weight}_{a+\varepsilon} > w_{\min}[\iota, \mathsf{weight}_{a+\varepsilon}](s) - 1)) > 0. \tag{7.2}$$

Notice that the following statements are equivalent

$$\begin{aligned} Pr_s^{\mathcal{M}}(\Box(-\mathsf{weight}_{a+\varepsilon} > w_{\min}[\iota, \mathsf{weight}_{a+\varepsilon}](s) - 1)) > 0 \\ Pr_s^{\mathcal{M}}(\Box(\mathsf{weight}_{a+\varepsilon} \leq -w_{\min}[\iota, \mathsf{weight}_{a+\varepsilon}](s))) > 0 \\ Pr_s^{\mathcal{M}}(\lozenge(\mathsf{weight}_{a+\varepsilon} > -w_{\min}[\iota, \mathsf{weight}_{a+\varepsilon}](s))) < 1. \end{aligned}$$

Hence, if we can show Equation (7.2) then Lemma 7.3 completes the argument.

Remember, π is an infinite path such that $\hat{\pi} \diamond \pi$ satisfies $\neg(\mathsf{goal}\,\mathcal{R}^\bullet \lozenge(\mathsf{ratio} > c))$. We can safely assume that π reaches a BSCC $\mathcal{C}$ with $\mathbb{E}_{\mathcal{C}}(\mathsf{ratio}) \leq c$ which (using the same argument as above for ϑ) implies $\mathbb{E}_{\mathcal{C}}(\mathsf{ratio}) \leq a$. Let $m > 0$ be the smallest position such that $\pi[m] \in \mathcal{C}$ and $\pi[\ldots m]$ contains a cycle with $\mathsf{ratio}(\vartheta) \leq c$. Thus, there exist $\hat{\pi}_1, \hat{\pi}_2 \in \mathsf{FinPaths}^{\mathcal{M}}$ and a weight_c-negative simple cycle ϑ such that $\pi[\ldots m] = \hat{\pi}_1 \diamond \vartheta \diamond \hat{\pi}_2$ and we can safely assume that every nonempty prefix $\rho \neq \hat{\pi}_1 \diamond \vartheta$ of $\hat{\pi}_1 \diamond \vartheta$ is a simple path. Further $\mathsf{weight}_c(\hat{\pi} \diamond \rho) < 0$ and hence as $\mathsf{ratio}(\hat{\pi} \diamond \rho) \in Q_{release}$ we have $\mathsf{weight}_{a+\varepsilon}(\hat{\pi} \diamond \rho) < 0$. Thus,

$$\mathsf{weight}_{a+\varepsilon}(\rho) = \mathsf{weight}_{a+\varepsilon}(\hat{\pi} \diamond \rho) - \mathsf{weight}_{a+\varepsilon}(\hat{\pi}) < -w_{min}[\iota, \mathsf{weight}_{a+\varepsilon}](s).$$

Notice that this yields $cdist[s, -\mathsf{weight}_{a+\varepsilon}](\pi[m], 0) = -\infty$. Thus, Lemma 4.35 yields

$$\begin{aligned} & Pr_s^{\mathcal{M}}(\Box(-\mathsf{weight}_{a+\varepsilon} > w_{\min}[\iota, \mathsf{weight}_{a+\varepsilon}](s) - 1))) > \\ & \quad Pr_s^{\mathcal{M}}(\Box(-\mathsf{weight}_{a+\varepsilon} > w_{\min}[\iota, \mathsf{weight}_{a+\varepsilon}](s))) > 0 \end{aligned}$$

and (as discussed above) completes the proof. $\square$

8 Conclusions

In this thesis, we considered several decision and computational problems related to weight and ratio objectives in annotated Markov chains. By stating graph-based characterizations relying only on BSCC anlyses and variants of standard graph algorithms, we showed that one can decide in polynomial-time, whether a ratio or weight objective under an ω-regular side constraint has to be fulfilled almost-surely or with positive probability. Further, using this decision procedure, we established efficient algorithms to exactly compute related quantiles. Whereas weight quantiles can be computed using a binary search over a finite and discrete interval, the ratio quantiles are more involved. For ratio quantiles, we presented an polynomial-time computation procedure, which first approximates the quantile value and then uses the continued-fraction method to retrieve the exact value out of a finite set of candidates. A comparison of the presented polynomial-time algorithm and the naive exponential-time computation of the candidate set is left for future work.

The presented techniques can be, to some extend, amended for events where instead of a single weight function, multiple weight functions are constrained [10]. Exploiting the EXPSPACE-completeness of the coverability problem for vector addition systems with states [11, 23] one can show that the following problem is complete for the complexity class EXPSPACE [22]: Given a Markov chain with weight functions $\mathsf{weight}_1, \ldots, \mathsf{weight}_d$ and $z_1, \ldots, z_d \in \mathbb{Z}$, do we have $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight}_1 > z_1) \wedge \ldots \wedge \Box(\mathsf{weight}_d > z_d)) > 0$? However, there are open problems still to be investigated, e.g., multi-weight problems containing arbitrary modalities or multi-ratio and multi-weight quantiles for which the reduction to vector addition systems with states presented in [22] is not applicable.

Quantitative ratio and weight problems are another area which is left for further work. It is known that already the quantitative $\Box$-weight decision problem, i.e., deciding $Pr_\iota^{\mathcal{M}}(\Box(\mathsf{weight} > z)) > p$ for a given $p \in [0, 1]$, is PosSLP-hard for unit-weight Markov chains [15, 9, 4]. Thus, efficient exact computation procedures for the corresponding quantitative quantiles cannot be expected. Exploiting the results of this thesis, one (expensive) way to compute quantitative weight quantiles is to perform a simple binary search applying well-known polynomial-space algorithms for analyzing probabilistic one-counter automata on the interval induced by the corresponding qualitative quantiles [8]. As ratio decision problems are reducible to weight decision problems, this approach can also be used to approximate quantitative ratio quantiles.

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Confirmation

I confirm that I independently prepared the thesis and that I used only the references and auxiliary means indicated in the thesis.