Group testing with geometric ranges

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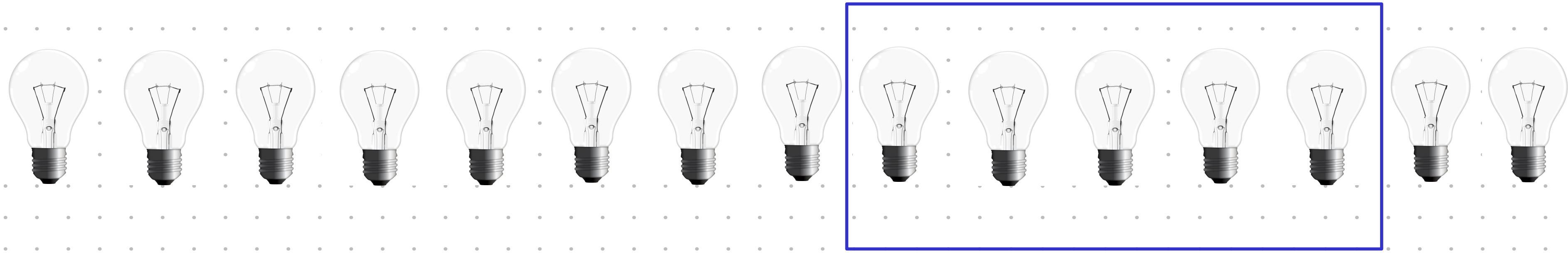
Group testing



test: are these all good?

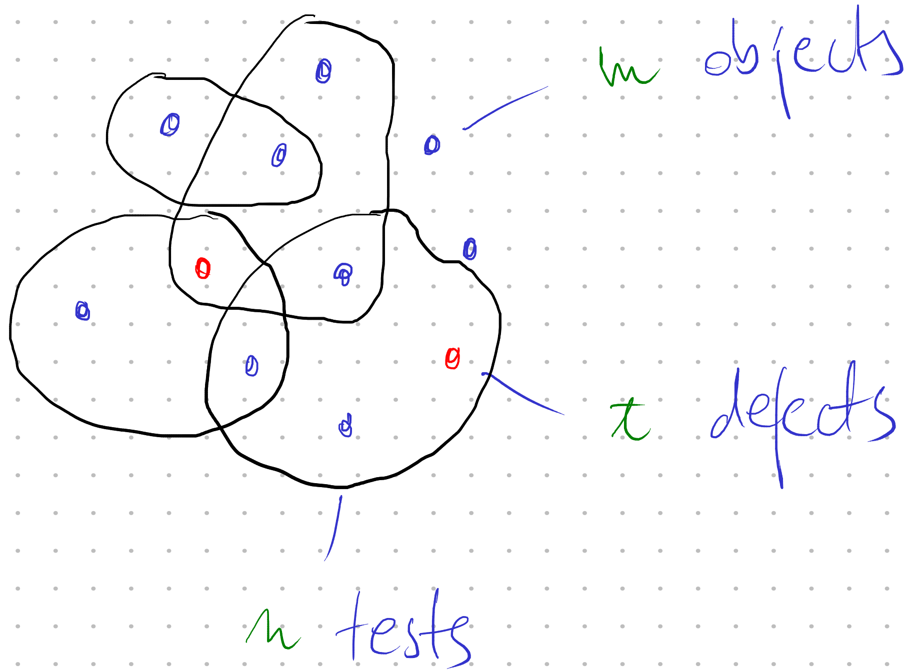
YES/NO

m objects

n tests (non-adaptive)

t defects

→ Can a given configuration of objects/tests identify t defective objects?



→ non-adaptive test configuration: can be viewed as a set system

→ Can a given configuration identify t defectives?

→ For given t, n what is largest m we can handle?

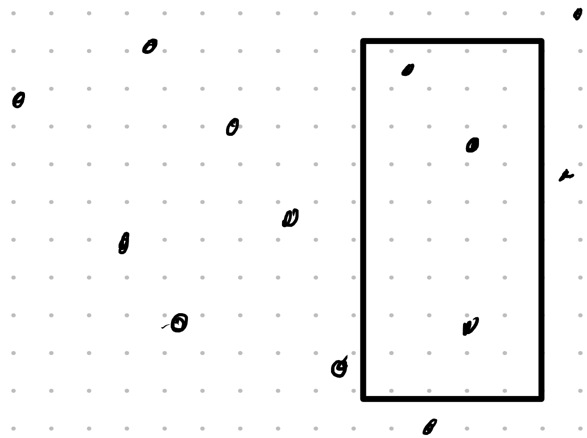
Classical group testing: for $t \in O(1)$, $m = 2^{\Theta(n)}$

(tests unrestricted)

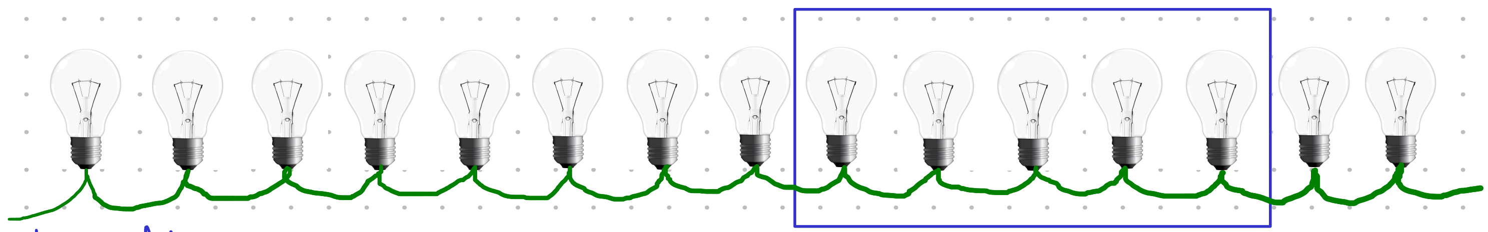
Our setting:

→ Objects are points in $\mathbb{R}^d$

→ Tests are rectangles in $\mathbb{R}^d$ (parameter ranges)



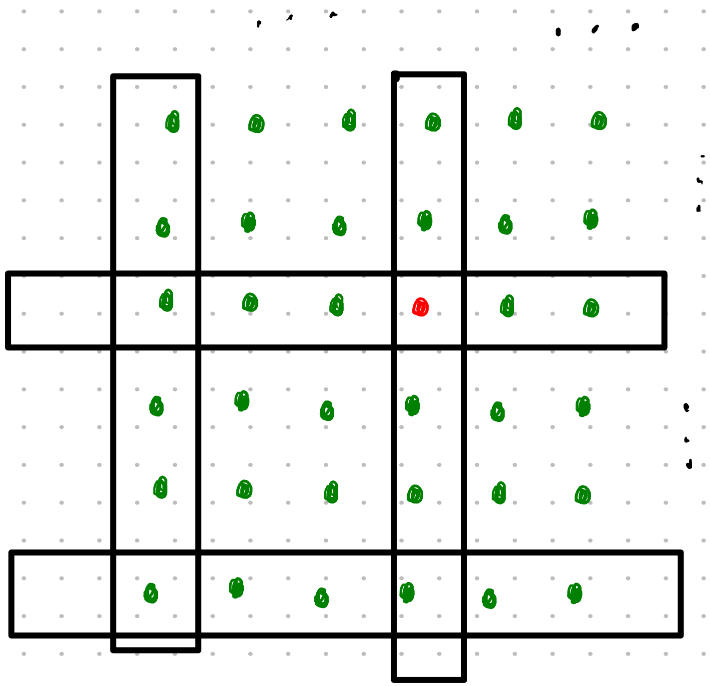
e.g. $\mathbb{R}^1$



m objects

need $n = \frac{m}{2}$ tests

e.g.



m objects

$t = 1$ defects

$n = 2\sqrt{m}$ tests

$\to$ How many objects (points) can we handle with n rectangle-tests, t defects, in $\mathbb{R}^d$?

$S_t^d(n)$: max # of points such that there exists a valid configuration

$S_t^d(n)$

Valid configuration:

Def. for every two sets A, B such that $|A| = |B| = t$

$\{$set of tests intersecting $A\} \neq \{$set of tests intersecting $B\}$

(can distinguish cases "A is defect set" and
"B is defect set")

$\rightarrow$ t-separable

$S_t^d(n)$ $\longrightarrow$ goal: understand asymptotic behavior.

Note: does not imply an efficient algorithm for identifying defects.

For that we have a stronger property $\to$ t-disjunct

Similar quantity: $\longrightarrow D_t^d(n)$

Obs. $D_t^d(n) \leq S_t^d(n)$

Goal: asymptotics of $S_t^d(n)$ (max # of points with
n rectangle-tests
t defects
d dimensions)

Recall combinatorial setting: $2^{\Theta(n)}$

Easy observations:

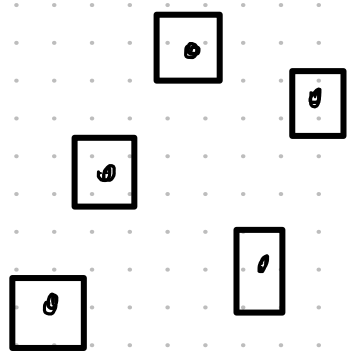
$$S_t^d(n) \geq S_{t+1}^d(n)$$

$$S_t^d(n) \leq S_t^{d+1}(n)$$

Trivial lower bound:

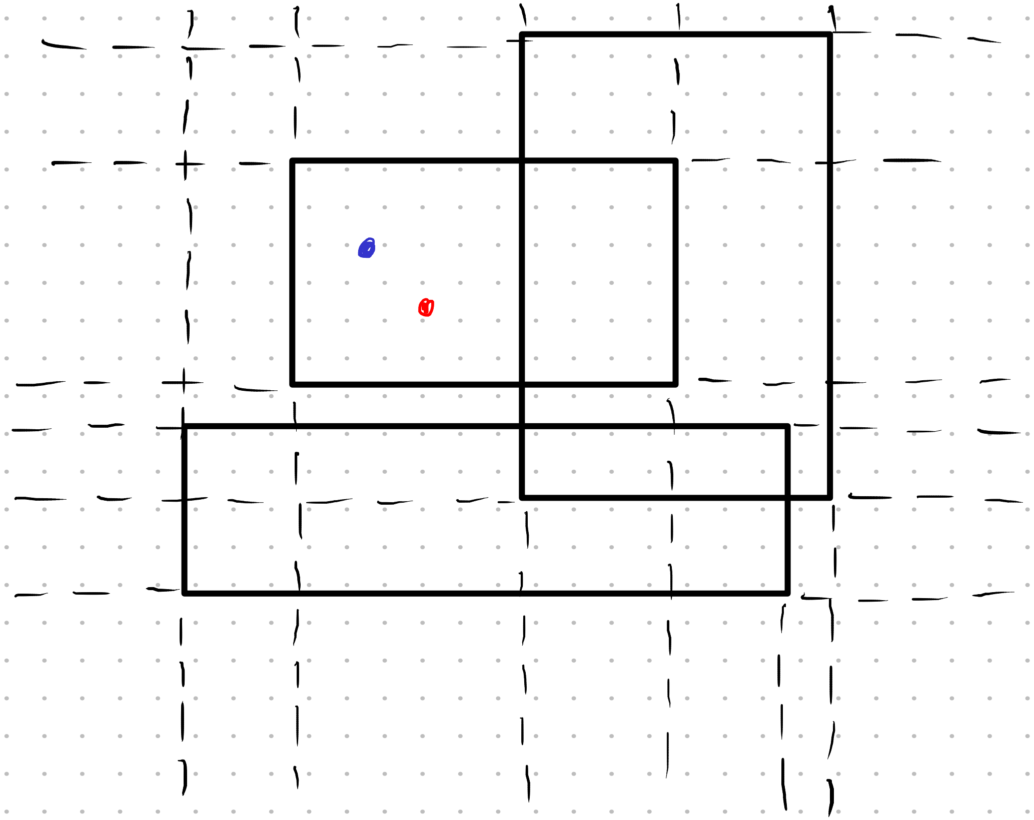
$$n \leq S_t^d(n)$$

→ n tests can always handle at least n points:



(Almost) Trivial upper bound:

$$S_t^d(n) \leq n^d$$



→ n rectangles define $(2n)^d$ cells

→ points in same cell cannot be distinguished

$$n \leq S_t^d(n) \leq n^d$$

→ polynomial instead of exponential

→ What is the exponent?

<u>Claim:</u>

$$S_1^d \sim n^d$$

$$S_t^1 \sim n$$

<u>Main result:</u> neither LB nor UB tight for $t, d \geq 2$.

Result:

$$n^{1+f(t,d)} \leq S_t^d(n) \leq n^{d-g(t,d)}$$

→ nontrivial improvements $f(\cdot), g(\cdot)$

→ gap grows with d, t

e.g. $n^{3/2} \leq S_2^2 \leq n^{5/3}$

→ few tight results

e.g. $D_2^2 \sim n^{3/2}$

Open: close gaps, find separations.

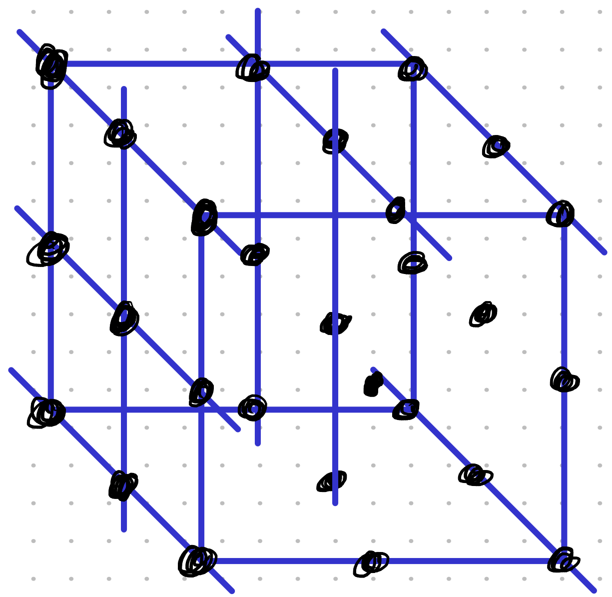
Lower bound sketch:

$$n^{1+2/t} \leq S_t^2(n)$$

→ need to show a concrete feasible configuration

Idea:

start with an abstract construction:



→ k^d grid points

→ $d \cdot k^{d-1}$ lines as tests

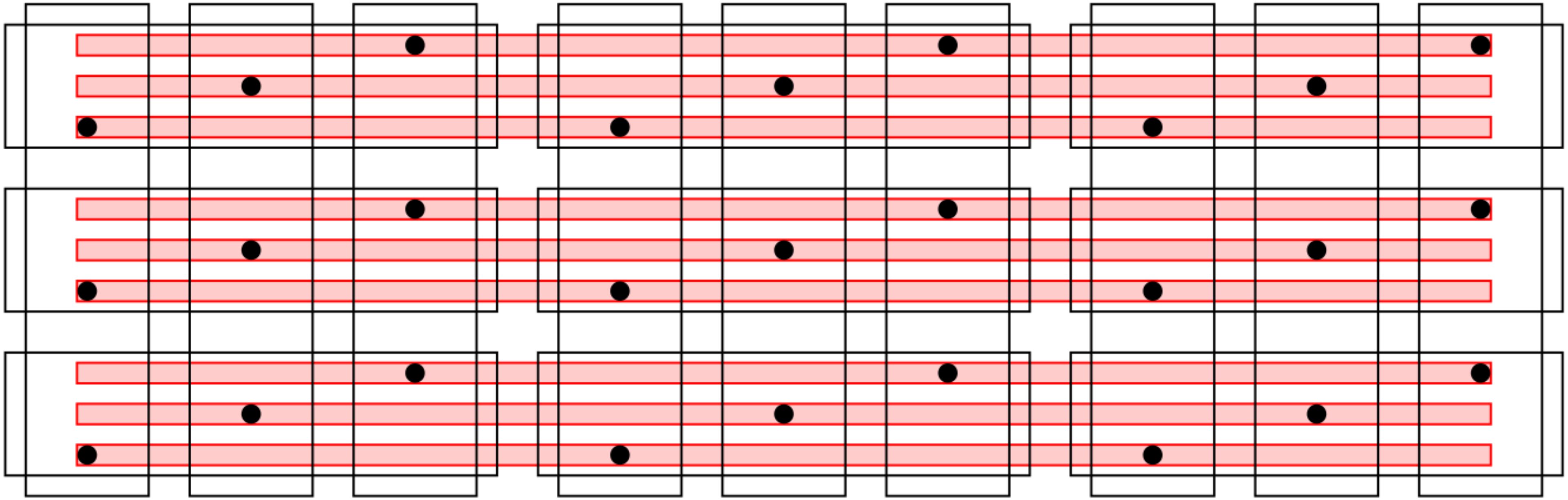
→ exact characterization:

$\sim 2d$-separable

→ Can "project" to equivalent rectangle-system in $\mathbb{R}^2$

→ can set k as needed

e.g. 3-separable system in $\mathbb{R}^2$

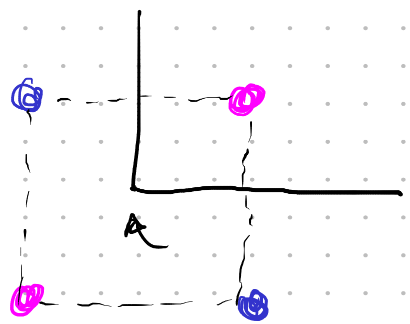


Upper bounds

e.g. $S_2^2(n) \in O(n^{5/3})$ (no valid configuration with more points)

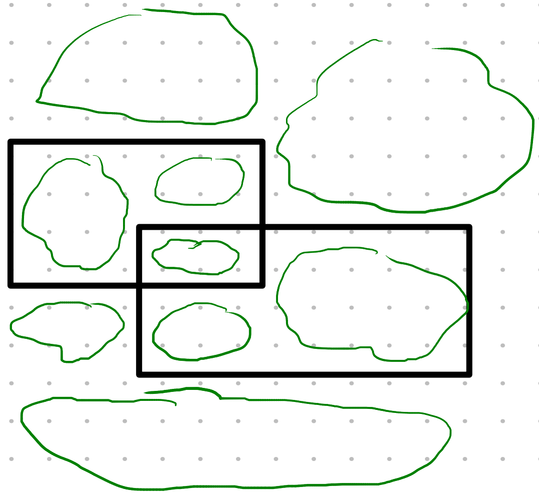
→ discretize to $4n \times 4n$ grid

→ Observe: if 4 points (objects) form a rectangle, must have some test corner inside



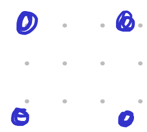
otherwise cannot distinguish diagonal pairs.

→ Observe: if 4 points (objects) form a rectangle, must have some test corner inside



Proof idea: → partition space into regions without rectangle corners

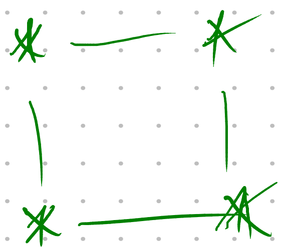
⇒ these regions cannot have 4 points forming rectangles



⇒ these regions cannot have too many points

⇒ overall we cannot have too many points ∎

Thm. [Kövári - Sós - Turán]
Any $n \times n$ matrix with at least $n^{3/2}$ $*$-entries must contain:



somewhere.

→ carefully control parameters

→ similar bounds by lifting other theorems from extremal combinatorics.

Conclusion

- possibility and limitations of group testing with rectangles
- n tests can handle n^c objects (as opposed to c^n)

- Lower bounds: concrete geometric constructions
- Upper bounds: impossibility via extremal combinatorics

Open:

- exact bounds elusive, large gaps remain
- asymptotic separations for different t, d?

THANKS!