Fixed-Point Cycles and Approximate EFX Allocations

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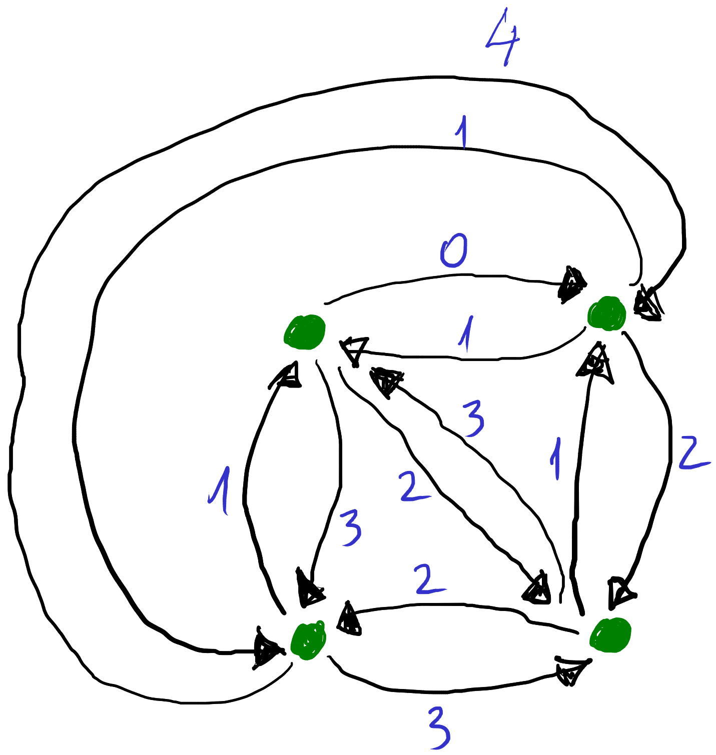
1. A "puzzle" on graphs:

→ bidirected, complete graph $\overleftrightarrow{K}_n$, edge-labels from $\{0, \dots, d-1\}$

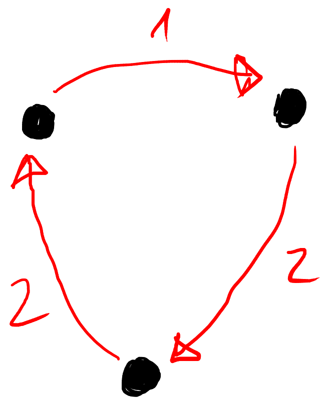
e.g. $\overleftrightarrow{K}_4$ and $d = 5$



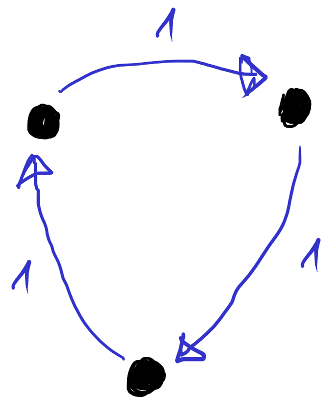
→ bidirected, complete graph $\overleftrightarrow{K}_n$, edge-labels from $\{0, \dots, d-1\}$

→ no simple cycle with zero sum $(\mathrm{mod}\ d)$.

e.g.



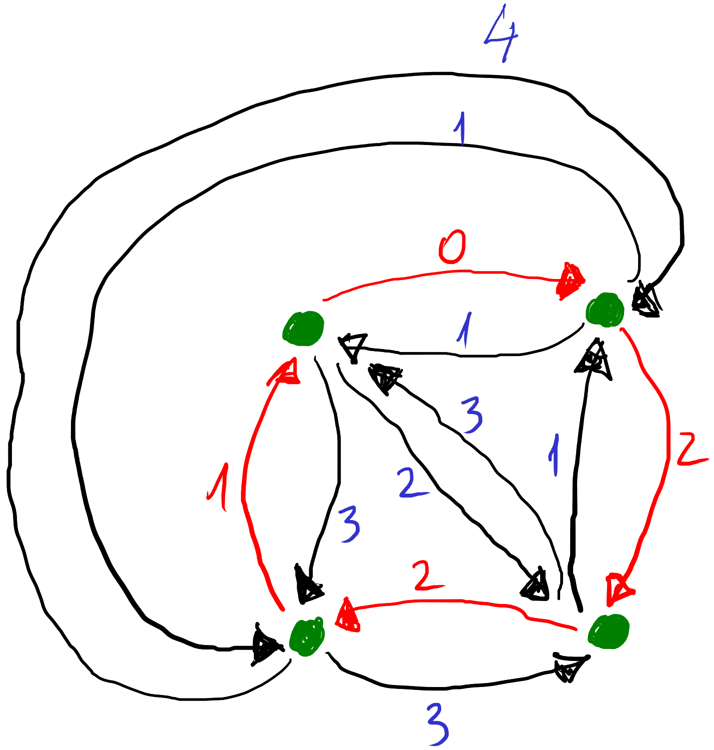
sum $= 0 \ (\mathrm{mod}\ 5)$



sum $= 3 \ (\mathrm{mod}\ 5)$

→ labeling is zero-sum-free

e.g. $d=5$



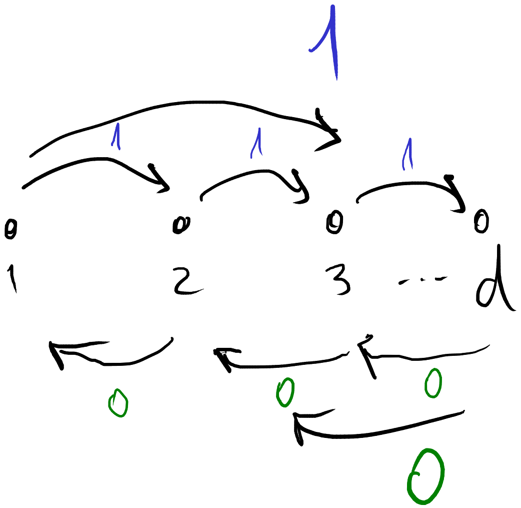
(not zero-sum-free)

Fix d.

Largest n for which we have a zero-sum-free labeling?

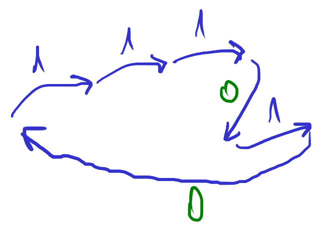
$$n = R(d)$$

Obs. $R(d) \geq d$



zero-sum-free

(each cycle has between 1 and $d-1$ right-edges)



Q. Is $R(d)$ finite?

Thm [Alon, Krivelevich, 2021]

$$R(d) \in O(d \log d)$$

$$R(d) \in O(d) \quad \text{if } d \text{ prime}$$

Thm [Mészáros, Steiner, 2021]

$$R(d) \leq 8d$$

Our result

$$R(d) \leq 2d-2$$

- easier proof through a broad generalization
- also holds for arbitrary groups $(G_d, +)$

Open:

$$d \leq R(d) \leq 2d-2$$

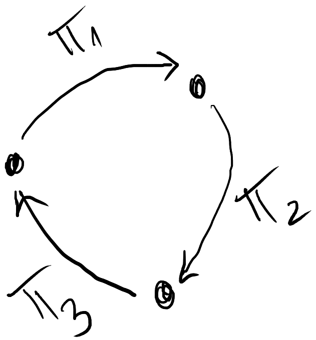
may be tight for some nice "algebraic" reason

2. Generalization to fixed-point-cycles:

→ bidirected, complete graph $\overleftrightarrow{K}_n$,

edges labeled with permutations $\pi : [d] \to [d]$

s.t. no simple cycle has fixed point.



$$\sigma = \pi_1 \circ \pi_2 \circ \pi_3$$

$$\sigma(x) \neq x$$

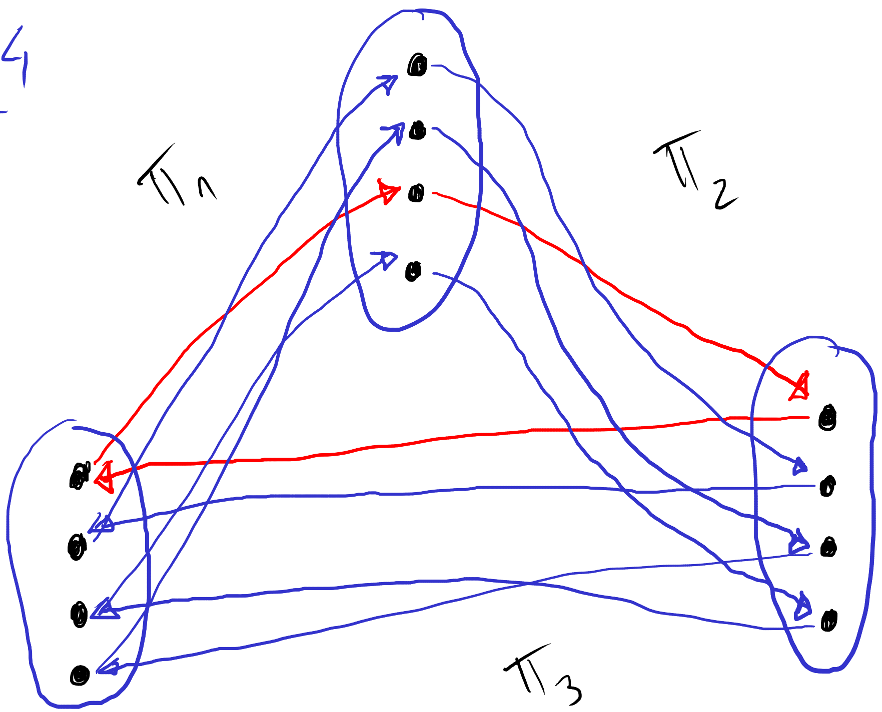
2. Generalization to fixed-point-cycles:

→ bidirected, complete graph $\overleftrightarrow{K}_n$,

edges labeled with permutations $\pi : [d] \to [d]$

s.t. no simple cycle has fixed point.

e.g. $\underline{d = 4}$



(has fixed point)

Fix d.

Largest n for which we have a fixed-point-free labeling?

$$n = R_p(d)$$

Obs. $d \le R(d) \le R_p(d)$

Permutation case is more general:

→ Take permutations of form $\pi_i(x) = x + i \pmod d$

fixed-point-cycle $\equiv$ zero-sum-cycle

Our result

$$R_p(d) \leq 2d - 2$$

(General result appears easier to prove, permutation-view suggests simple inductive argument)

3. Try something even more general:

→ bidirected, complete graph $\overleftrightarrow{K}_n$,

edges labeled with functions $\pi : [d] \to [d]$

s.t. no simple cycle has fixed point.

max n for which possible:

$n = R_\rho(d)$

Obs. $d \leq R(d) \leq R_p(d) \leq R_f(d)$

(permutations $\subseteq$ functions)

Actually, $R_f(d)$ has been studied before:

[Chaudhury, Garg, Mehlhorn, Mehta, Misra, 2021]

Is $R_f(d)$ even finite?

Yes, Ramsey-thm $\Longrightarrow$ $R_f(d) \leq 2^{2^{2^{O(d)}}}$

Thm [Chaudhury, Garg, Mehlhorn, Mehta, Misra, 2021]

$$R_f(d) \in O(d^4)$$

Our result

$$R_f(d) \in d^{2+o(1)}$$

Independent work [Akrami et al. 2022]

$$R_f(d) \in O(d^2)$$

Summary

$$d \leq R(d) \leq R_p(d) \leq R_f(d) \leq O(d^2)$$

LB could be tight for all

verified for small d via SAT - solver

Why study $R_f(d)$?

It relates to a central problem of fair allocation.

→ n agents want to partition m goods (each have their own valuations)

e.g. inheritance, frequency allocations, etc.

→ Envy-free allocation

not always possible

→ EFX: Envy-free up to one good [Caragiannis et al.]

Open: Is EFX always possible?

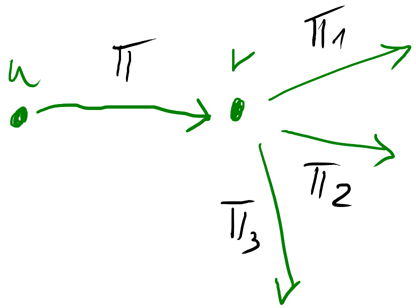
Thm [Chaudhury, Garg, Mehlhorn, Mehta, Misra, 2021]

If $R_f(d) \le O(d^c)$, then there always exists a $(1+\varepsilon)$-EFX allocation with $O\left(n^{\frac{c}{c+1}}\right)$ goods not allocated.

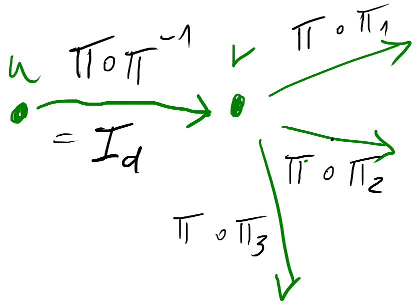
Corollary

$R_f(d) \in d^{2+o(1)} \Rightarrow$ only $O(n^{2/3})$ unallocated goods.

Proof idea (permutations)



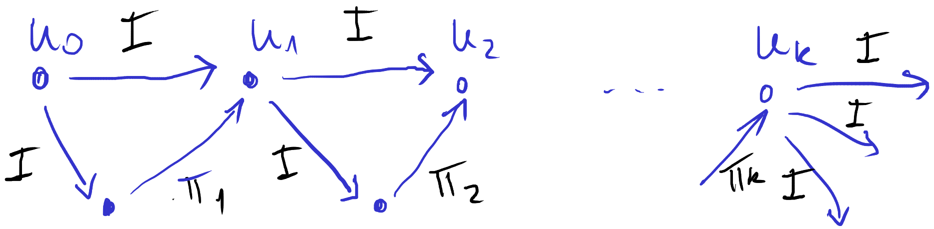
transform to



"shifting"

(cycles unchanged)

→ Build a chain step-by-step:

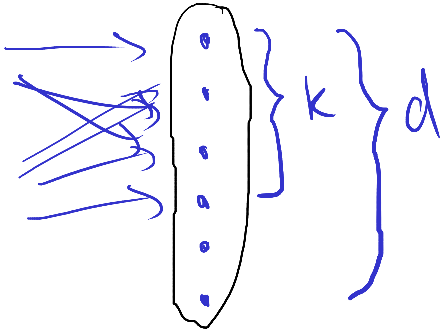


R_k : reachable set from 1 in u_0

- if $R_k = [d]$, done
- o.w. extend chain s.t. $R_{k+1} \supset R_k$
- o.w. $R_k \leftrightarrow R_k$
  $[d]-R_k \leftrightarrow [d]-R_k$, recurse

Proof idea (functions)

Vertex is k-compressed if all incoming edges only map to same k-subset of values



Obs. If more than $R_f(k)$ vertices k-compressed, we can find fixed-point cycle

Strategy. Transform paths (reversibly) s.t. we further compress vertices. (Preserves existence of fixed-point cycle)

Summary

$$\boxed{\text{Extremal combinatorics [zero-sum problems] meets Algorithmic game theory [fair allocations]}}$$

Thm. $d \leq R(d) \leq R_p(d) \leq R_f(d) \lesssim d^{2+o(1)}$

Conjecture. $d = R(d) = R_p(d) = R_f(d)$

Open. Extend other classical results from zero-sums to fixed-point cycles